LESS STRESS MORE SUCCESS

CONTENTS

Please note:

The exam questions marked by the symbol exam Q in this book are selected from the following:

1. SEC exam papers
2. Sample exam papers
3. Original and sourced exam-type questions

1 Number Systems

aims

- To learn what the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ and $\mathbb{R}$ represent.
- To be familiar with prime numbers, factors and the fundamental theorem of arithmetic.
- To be able to find LCM and HCF as required.
- To have a clear understanding of and be able to complete calculations using the order of operations.

Number sets

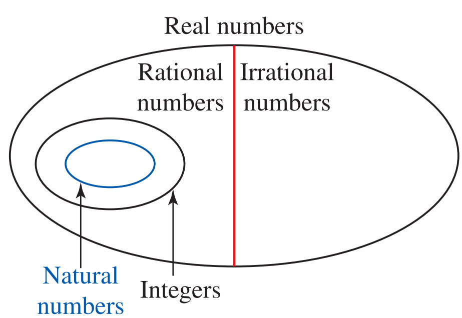
Four of the five number sets required on our course are to be found in the booklet of formulae and tables. It gives us:

	Number sets
$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \cdots\}$	Natural numbers
$\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$	Integers
$\mathbb{Q} = \left\{\frac{p}{q} \middle\| p \in \mathbb{Z}, \quad q \in \mathbb{Z}, \quad q \neq 0\right\}$	Rational numbers
$\mathbb{R}$	Real numbers

The set not given in the tables is the set of irrational numbers, represented by $\mathbb{R}\backslash\mathbb{Q}$.

We meet the set of irrational numbers later in this chapter.

A Venn diagram of the number system looks like this:



Natural numbers $\mathbb{N}$

The positive whole numbers 1, 2, 3, 4, 5 . . . are also called the counting numbers. The dots indicate that the numbers go on forever and have no end (infinite).

exam Q

Give two reasons why $-7 \cdot 3$ is not a natural number.

Solution

Reason 1: It is a negative number.

Reason 2: It is not a whole number (it is a decimal).

Factors (divisors)

key point

The factors of any whole number are the whole numbers that divide exactly into the given number, leaving no remainder.

- 1 is a factor of every number.
- Every number is a factor of itself.

Example

Find the factors of 18.
Find the factors of 45.
Hence find the highest common factor of 18 and 45.

Solution

18	45
1×18	1×45
2×9	3×15
3×6	5×9

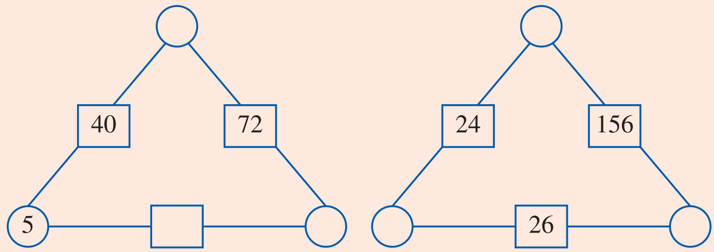
The common factors are 1, 3 and 9.
$\therefore$ The highest common factor of 18 and 45 is 9.

key point

The highest common factor (HCF) of two or more numbers is the largest factor that is common to each of the given numbers.

Example

In these productogons, the number in each square is the product of the numbers in the circles on each side of it. Find the missing numbers in each of these productogons.

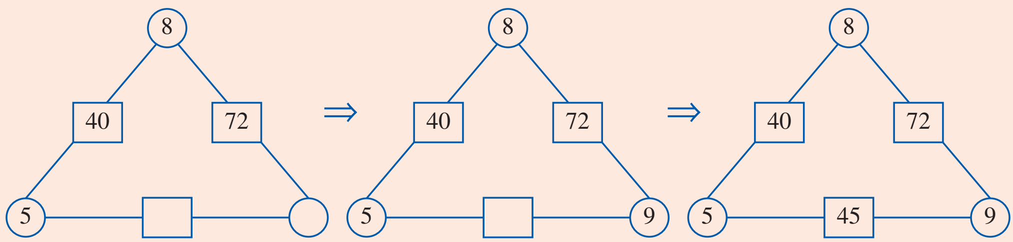


Solution

key point

The use of the word productogon in the question indicates we use multiplication. This is because product means multiply.

This first one is very straightforward.

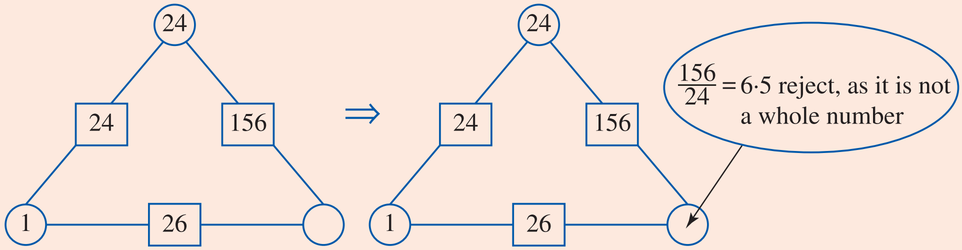


The second one is more challenging.
The method of trial and improvement (yes, guesswork!) is used here. Of the three given numbers, 24 and 26 would seem to have the easiest factors for us to find.

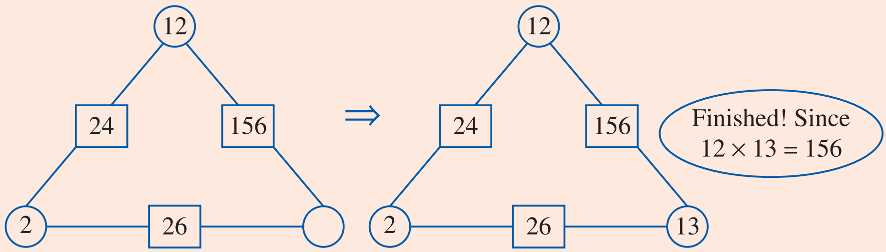
24	26
1×24	1×26
2×12	2×13
3×8	
4×6	

This indicates the bottom left-hand number is either 1 or 2, as they are the only factors common to both 24 and 26.

Let's put 1 into the bottom left-hand corner:



Now let's put 2 into the bottom left-hand corner:



Prime numbers

key point

A prime number is a whole number greater than 1 that has only two factors, 1 and itself.

The first 12 prime numbers are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 37.

There is an infinite number of prime numbers.

key point

Numbers that have more than two factors are called composite numbers.

The first 12 composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20 and 21. There is an infinite number of composite numbers.

exam focus

The fundamental theorem of arithmetic states that any whole number greater than 1 can be written as the product of its prime factors in a **unique** way. This will underpin many exam questions on number theory.

Prime factors

Any number can be expressed as a product of prime numbers. To express the number 180 as a product of its prime numbers, first divide by the smallest prime number that will divide exactly into it.

The smallest prime number 2	:	2	180
The smallest prime 2 again	:	2	90
The smallest prime 3	:	3	45
The smallest prime 3 again	:	3	15
The smallest prime 5	:	5	5
			1

So 180 expressed as a product of primes is $2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$.

Example

For security, a credit card is encrypted using prime factors. A huge number is assigned to each individual card and it can only be verified by its prime factor decomposition. Find the 10-digit natural number which is assigned to the following credit cards whose prime factor decomposition is

(i) $2 \times 3 \times 11 \times 13 \times 17^2 \times 19^3$

(ii) $2^7 \times 3^2 \times 5^2 \times 7^3 \times 23 \times 31$

Solution

By calculator (i) 1700771358

(ii) 7043299200

Example

Geppetto makes wooden puppets. He has four lengths of wood which he wants to cut into pieces, all of which must be the same length and be as long as possible. The lengths of the four pieces of wood are 315 cm, 357 cm, 210 cm and 252 cm.

(i) Express each of the four lengths as a product of primes.

(ii) Hence, calculate what length each piece should be and how many pieces he will have.

Solution

(i)

$$\begin{array}{r|l} 3 & 315 \\ 3 & 105 \\ 5 & 35 \\ 7 & 7 \\ & 1 \end{array} \qquad \begin{array}{r|l} 3 & 357 \\ 7 & 119 \\ 17 & 17 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 210 \\ 3 & 105 \\ 5 & 35 \\ 7 & 7 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 252 \\ 2 & 126 \\ 3 & 63 \\ 3 & 21 \\ 7 & 7 \\ & 1 \end{array}$$

$3^2 \times 5 \times 7$ $\qquad$ $3 \times 7 \times 17$ $\qquad$ $2 \times 3 \times 5 \times 7$ $\qquad$ $2^2 \times 3^2 \times 7$

(ii) By observation of the four 'products of primes' above:

The highest common factor (HCF) is given by $3 \times 7 = 21$.

Hence, each piece of wood should be 21 cm long.

The number of pieces is given by

$$\frac{315}{21} + \frac{357}{21} + \frac{210}{21} + \frac{252}{21}$$

$$= 15 + 17 + 10 + 12$$

$$= 54$$

key point

3 × 7 is common to all four lengths.

Integers $\mathbb{Z}$

Negative numbers are numbers below zero. Positive and negative **whole** numbers including 0 are called integers.

Integers can be represented on a number line:

Integers to the right of zero are called **positive integers**.

Integers to the left of zero are called **negative integers**.

Example

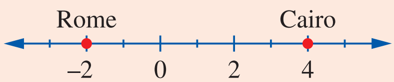
At midnight on Christmas Eve the temperatures in some cities were as shown in the table.

(i) Which city recorded the
  (a) Lowest temperature
  (b) Highest temperature?
(ii) List the temperatures from coldest to hottest.
(iii) Which cities had a temperature difference of 6°C?
(iv) What is the difference in temperature between
  (a) Dublin and Moscow
  (b) Cairo and Dublin?

New York	2°C
Rome	−2°C
Dublin	−1°C
Moscow	−20°C
Cairo	4°C

Solution

(i) (a) Lowest temperature, −20°, in Moscow
  (b) Highest temperature, 4°, in Cairo
(ii) −20, −2, −1, 2, 4
(iii)



Rome and Cairo had a difference of 6°C.

(iv) (a) Dublin and Moscow $= -1 - (-20) = -1 + 20 = 19°$
  (b) Cairo and Dublin $= 4 - (-1) = 4 + 1 = 5°$

Multiplication and division of two integers

The following two rules are applied to the multiplication or division of two integers.

1. If the signs are the same, then the answer will be positive.

   e.g. $\dfrac{-10}{-2} = +5; \quad (-10)(-2) = +20; \quad \dfrac{+10}{+2} = +5$

2. If the signs are different, then the answer will be negative.

   e.g. $\dfrac{-10}{+2} = -5; \quad (+10)(-2) = -20; \quad \dfrac{+10}{-2} = -5$

Example

Find the missing number in each box.

(i) $\square \times 5 = -10$ (ii) $8 \times \square = -24$

(iii) $-12 \div \square = 4$ (iv) $\square \div -9 = -4$

Solution

(i)
$$\square \times 5 = -10$$
$$\square = \frac{-10}{5}$$
$$\square = -2$$

(ii)
$$8 \times \square = -24$$
$$\square = \frac{-24}{8}$$
$$\square = -3$$

(iii)
$$-12 \div \square = 4$$
$$\frac{-12}{\square} = 4$$
$$-12 = 4\square$$
$$\frac{-12}{4} = \square$$
$$-3 = \square$$

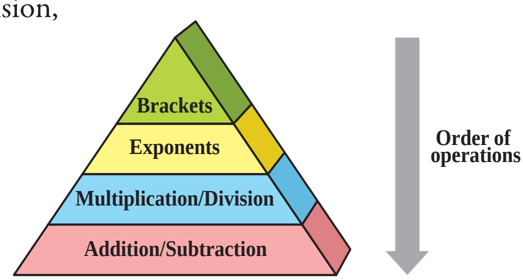
(iv)
$$\square \div -9 = -4$$
$$\frac{\square}{-9} = -4$$
$$\square = (-4)(-9)$$
$$\square = 36$$

Order of operations

A memory aid for the order of operations is BEMDAS (brackets, exponents, multiplication and division, addition and subtraction).

key point

Exponent(s) = Power(s) = Index (indices)



Example

Calculate: (i) $8 + 108 \div -9$ (ii) $10 \times 4 - 30 \div 6 + 19$

Solution

(i) $8 + 108 \div -9$

$= 8 - 12$ Division

$= -4$ Subtraction

(ii) $10 \times 4 - 30 \div 6 + 19$

$= 40 - 30 \div 6 + 19$ Multiplication

$= 40 - 5 + 19$ Division

$= 59 - 5$ Addition

$= 54$ Subtraction

Example

Calculate $4(5-3)^2 + 24 \div (6-2)$.

Solution

$4(5-3)^2 + 24 \div (6-2)$	
$= 4(2)^2 + 24 \div 4$	Brackets
$= 4(4) + 24 \div 4$	Exponents/powers
$= 16 + 24 \div 4$	Multiplication
$= 16 + 6$	Division
$= 22$	Addition

exam focus

Remember to confirm your answer with a calculator.

Fractions

A fraction is written as two whole numbers, one over the other, separated by a bar.

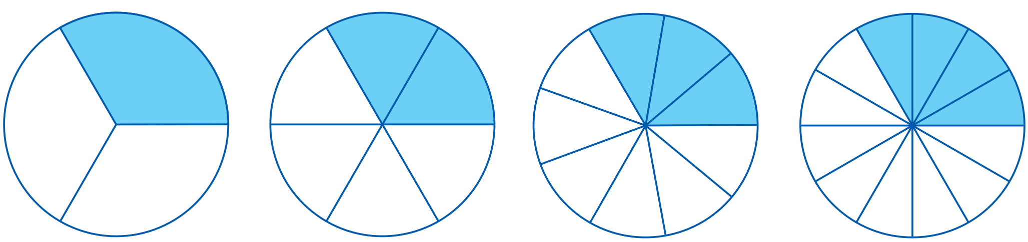
key point

$$\text{Fraction} = \frac{\text{Numerator}}{\text{Denominator}}$$

Equivalent fractions are fractions that are equal. For example:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$$

This can be shown on a diagram where the same proportion is shaded in each circle.



$\frac{1}{3} = \frac{2}{6}$ $\frac{3}{9} = \frac{4}{12}$

key point

The order of operations, BEMDAS, for fractions is the same as for integers.

Example

Bren is trying to subtract $\frac{1}{5}$ from $\frac{7}{8}$.

His attempt is shown here: $\frac{7}{8} - \frac{1}{5} = \frac{6}{3} = 2$

(i) Explain what Bren has done wrong.

(ii) Write out the correct solution.

Solution

(i) It seems that Bren has subtracted top from top and bottom from bottom.

key point

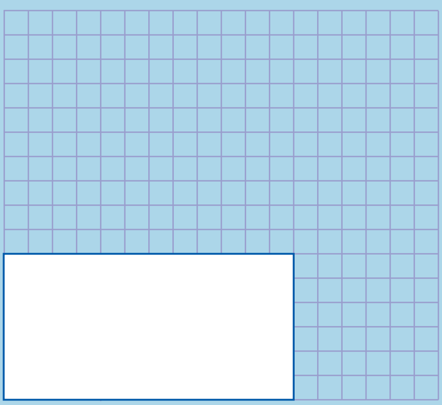
For subtraction or addition of fractions, we must find a common denominator.

(ii) $\frac{7}{8} - \frac{1}{5} \Rightarrow$ common denominator $= 8 \times 5 = 40$

$$\text{Then } \frac{7}{8} - \frac{1}{5} = \frac{(5)(7) - (8)(1)}{40} = \frac{35 - 8}{40} = \frac{27}{40}$$

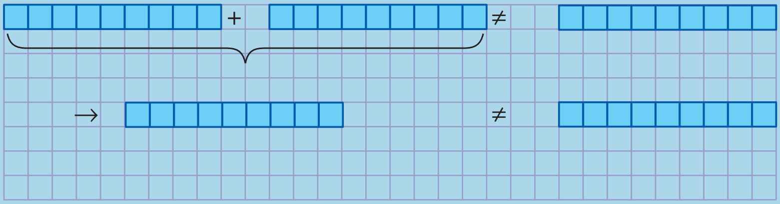
exam Q

(i) The diagram below shows three-fifths of a rectangle. Complete the rectangle on the grid.



(ii) By shading appropriate sections of the strips below, show that

$$\frac{1}{3} + \frac{2}{6} \neq \frac{3}{9}$$



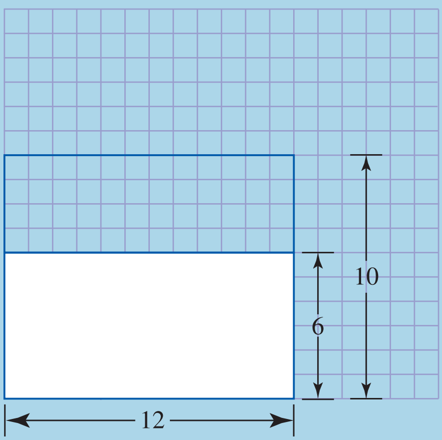
Solution

(i) By counting the rows (6) and the columns (12), the area of the given rectangle equals $6 \times 12 = 72$ square units.

This tells us that $\frac{3}{5}$ of the rectangle $= 72$ square units.

$\Rightarrow \frac{1}{5}$ of the rectangle $= \frac{72}{3} = 24$ square units.

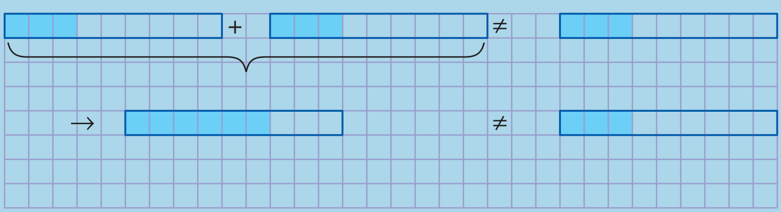
We conclude the full rectangle $= 24 \times 5 = 120$ square units.



exam focus

Many candidates simply counted 6 units (boxes) in height and since $\frac{6}{10} = \frac{3}{5}$ they wrote down the height of the full rectangle as 10 units (boxes) (see diagram).

(ii) $\frac{1}{3} + \frac{2}{6} \neq \frac{3}{9}$



key point

$$\frac{2}{6} = \frac{1}{3}$$

exam Q

(a) (i) Write the numbers **3**, **9**, and **25** into the three empty boxes shown to make the mathematical statement true. Use each number only once.

$$\frac{\square}{5} + \frac{\square}{\square} = \frac{24}{25}$$

(ii) Write the numbers **3**, **5**, **9**, and **25** into the empty boxes shown so that the difference between the two fractions is as large as possible. Use each number only once.

$$\frac{\square}{\square} - \frac{\square}{\square}$$

(b) A positive whole number has exactly 4 factors. One of the factors is 9. Work out the number.

Solution

(a) (i) The result of $\frac{24}{25}$ is less than one, so both fractions on the left side must both be less than one. This means that the number 3 must go above the 5. And for the second fraction, the 9 must be above the 25.You can verify your answer using your calculator.

$$\frac{3}{5} + \frac{9}{25} = \frac{24}{25}$$

(ii) To make the difference as large as possible, the first fraction needs to be as large as possible and the second fraction needs to be as small as possible. To make a fraction large – make the numerator big and the denominator small. To make a fraction small – make the numerator small and the denominator big. To make the difference as large as possible, we need to arrange the numbers in the fractions as shown:

$$\frac{25}{3} - \frac{5}{9}$$

(b) Since one of the factors of the required number is 9, we know that the number we are looking for is a multiple of 9. We need to examine the multiples of 9 and see how many factors they have. The first one we find that has 4 factors will be a correct answer.

Multiple	Factors	Number of factors	
9	1, 9	2	✗
18	1, 2, 3, 6, 9, 18	6	✗
27	1, 3, 9, 27	4	✓

So, the answer is 27.

exam focus

Fractions (rational numbers, $\mathbb{Q}$) are the third set of numbers listed in the booklet of formulae and tables. The set $\mathbb{Q}$ contains all the integers $\mathbb{Z}$, which in turn contains all the natural numbers $\mathbb{N}$. You must know this.

The Venn diagram at the beginning of this chapter should help you understand this exam focus.

Irrational numbers

The word 'irrational' literally means 'no ratio'. Numbers which cannot be written as simple fractions are called irrational numbers (cannot be written as one integer divided by another integer). As decimals, they never repeat or terminate.

Using your calculator to evaluate $\sqrt{3}$ and π gives:

$$\sqrt{3} = 1{\cdot}732050808 \qquad \text{(irrational, never repeats or terminates)}$$
$$\pi = 3{\cdot}141592654\ldots \qquad \text{(irrational, never repeats or terminates)}$$

The popular approximation of $\pi = \dfrac{22}{7} = 3{\cdot}142857143\ldots$ is close but not accurate.

We use the set notation $\mathbb{R}\backslash\mathbb{Q}$ for irrational numbers.

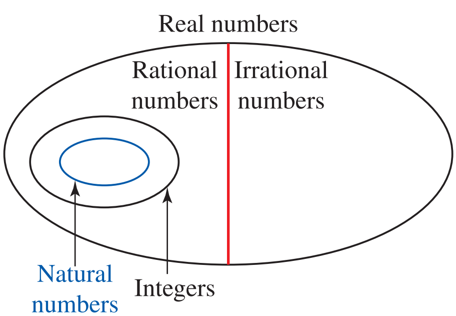
key point

A rational number cannot be an irrational number and an irrational number cannot be a rational number.

Real numbers $\mathbb{R}$

When rational numbers and irrational numbers are joined together, they form a set of numbers called the real numbers $\mathbb{R}$.

The following Venn diagram summarises the number system.



key point

Natural numbers are a subset of the integers. The integers are a subset of the rational numbers.

exam Q

(i) The columns in the table below represent the following sets of numbers:

Natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), irrational numbers ($\mathbb{R}\backslash\mathbb{Q}$) and real numbers ($\mathbb{R}$).

Complete the table by writing either 'Yes' or 'No' into each box, indicating whether each of the numbers $\sqrt{5}$, 8, -4, $3\frac{1}{2}$, $\frac{3\pi}{4}$ is or is not an element of each.

(One box has already been filled in. The 'Yes' indicates that the number 8 is an element of the set of real numbers, $\mathbb{R}$.)

Number/Set	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}\backslash\mathbb{Q}$	$\mathbb{R}$
$\sqrt{5}$					
8					Yes
-4					
$3\frac{1}{2}$					
$\frac{3\pi}{4}$					

(ii) In the case of $\sqrt{5}$, explain your choice in relation to the set of irrational numbers ($\mathbb{R}\backslash\mathbb{Q}$) (i.e. give a reason for writing either 'Yes' or 'No').

Solution

(i)

Number/Set	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}\backslash\mathbb{Q}$	$\mathbb{R}$
$\sqrt{5}$	No	No	No	Yes	Yes
8	Yes	Yes	Yes	No	Yes
-4	No	Yes	Yes	No	Yes
$3\frac{1}{2}$	No	No	Yes	No	Yes
$\frac{3\pi}{4}$	No	No	No	Yes	Yes

key point

8 is the only natural number, $\mathbb{N}$, in the question.

8 and -4 are the only integers, $\mathbb{Z}$.

8, -4 and $3\frac{1}{2}$ are all rational numbers, $\mathbb{Q}$.

$\sqrt{5}$ and $\frac{3\pi}{4}$ are both irrational numbers, while all the numbers are real, $\mathbb{R}$.

(ii) $\sqrt{5} = 2{\cdot}236067977\ldots$ Since this decimal never repeats or terminates, it is a real number but cannot be written as a fraction.

exam focus

Both parts of this question were very badly answered by the vast majority of candidates. As a result, the total marks awarded was 5 marks with 3 marks for **any one** correct answer.

Remember, stick to your time budget and always write something in the space provided.

18 Classroom-Based Assessments (CBAs)

aims

- ☐ To become familiar with the four elements of assessment for Junior Cycle Mathematics.
- ☐ To be familiar with the details of the Classroom-Based Assessment 1.
- ☐ To be able to understand and apply the Problem-Solving Cycle.
- ☐ To be familiar with the criteria of quality for assessment.
- ☐ To understand the four descriptors for the CBA and the criteria associated with each descriptor.
- ☐ To understand the steps involved in starting your investigation and examining a menu of suggestions for investigation.
- ☐ To be familiar with the procedure involved with how to carry out a mathematical investigation.
- ☐ To be able to use the checklist provided to ensure that you haven't missed any key elements in your investigation.

Introduction

As mentioned in the Introduction chapter of this book, your assessment in Junior Cycle Mathematics consists of four elements.

1. **Classroom-Based Assessment 1 (CBA 1)**

   This is a mathematical investigation and it is carried out during your second year of the three-year Junior Cycle. **CBA 1 is covered in this chapter.**

2. **Classroom-Based Assessment 2 (CBA 2)**

   This is a statistical investigation and it is carried out during your third year of the three-year Junior Cycle. **CBA 2 is covered in *Less Stress More Success Maths Book 2*.**

3. **Assessment Task**

   This is a written assignment and it is carried out during your third year of the three-year Junior Cycle, after you have completed CBA 2.

4. **Written exam paper**

   This is a 2-hour written exam and it take place at the end of third year, with the rest of your written exams.

CBA 1: Mathematical Investigation

The investigation is an opportunity for you to show that you can apply Mathematics to an area that interests you. Your teacher will give you a timetable and deadline for submitting your investigation.

The details of the investigation are as follows:

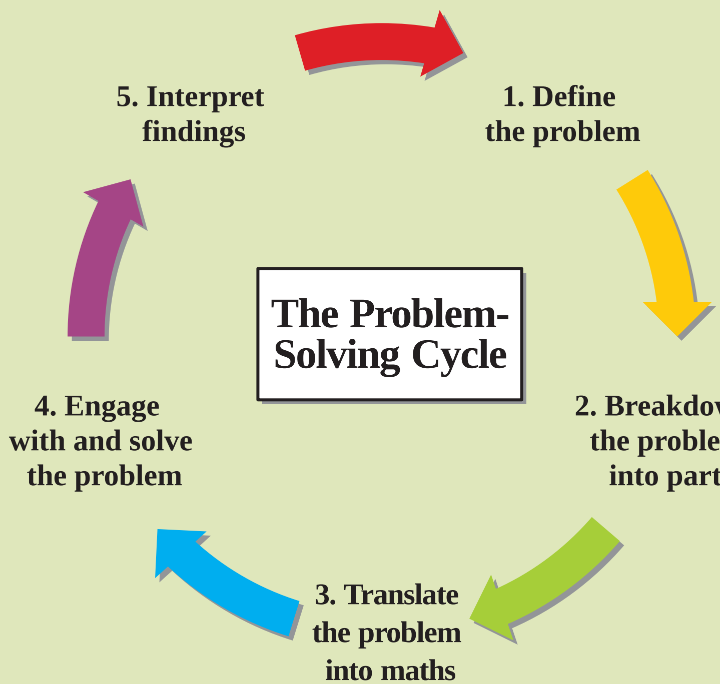
Format: A report may be presented in a wide range of formats.
Preparation: A student will, over a three-week period in second year, follow the Problem-Solving Cycle to investigate a mathematical problem.

The Problem-Solving Cycle is as follows:

1. Define a problem
2. Decompose it into manageable parts and/or simplify it using appropriate assumptions
3. Translate the problem to mathematics, if necessary
4. Engage with the problem and solve it, if possible
5. Interpret any findings in the context of the original problem

key point

You must be familiar with the Problem-Solving Cycle:



CBA 1: Assessment criteria and four descriptors

The investigation is assessed by the class teacher. A student will be awarded one of the following categories of achievement:

- Yet to meet expectations
- In line with expectations
- Above expectations
- Exceptional

Assessment criteria

A good investigation should be clear and easily understood by one of your fellow classmates (peers) and self-explanatory all of the way through. The criteria are split into four areas A, B, C and D:

A. Defining the problem statement
B. Finding a strategy or translating the problem to mathematics
C. Engaging with the mathematics to solve the problem
D. Interpreting and reporting

Linking the criteria with the four categories of achievement (descriptors)

A. Defining the problem statement

Criteria	Achievement
Uses a given problem statement and with guidance breaks the problem down into steps	Yet to achieve expectations
With guidance poses a problem statement, breaks the problem down into manageable steps and simplifies the problem by making assumptions, if appropriate	In line with expectations
With limited guidance poses a problem statement and clarifies/simplifies the problem by making reasonable assumptions, where appropriate	Above expectations
Poses a concise problem statement and clarifies and simplifies the problem by making justified assumptions, where appropriate	Exceptional

B. Finding a strategy or translating the problem to mathematics

Criteria	Achievement
Uses a given strategy	Yet to achieve expectations
Chooses an appropriate strategy to engage with the problem	In line with expectations
Justifies the use of a suitable strategy to engage with the problem and identifies any relevant variables	Above expectations
Develops an efficient justified strategy and evaluates progress towards a solution where appropriate; conjectures relationship between variables where appropriate	Exceptional

C. Engaging with the mathematics to solve the problem

Criteria	Achievement
Records some observations/data and follows some basic mathematical procedures	Yet to achieve expectations
Records observations/data and follows suitable mathematical procedures with minor errors; graphs and/or diagrams/words are used to provide insights into the problem and/or solution	In line with expectations
Records observations/data systematically, suitable mathematical procedures are followed, and accurate mathematical language, symbolic notation and visual representations are used; attempts are made to generalise any observed patterns in the solution/observation	Above expectations
Mathematical procedures are followed with a high level of precision, and a justified answer is achieved; solution/observations are generalised and extended to other situations where appropriate	Exceptional

D. Interpreting and reporting

Criteria	Achievement
Comments on any solution	Yet to achieve expectations
Comments on the reasonableness of the solution where appropriate and makes a concrete connection to the original question, uses everyday familiar language to communicate ideas	In line with expectations
Checks reasonableness of solution and revisits assumptions and/or strategy to iterate the process, if necessary, uses formal mathematical language to communicate ideas and identifies what worked well and what could be improved	Above expectations
Deductive arguments used and precise mathematical language and symbolic notation used to consolidate mathematical thinking and justify decisions and solutions; strengths and/or weaknesses in the mathematical representation/solution strategy are identified	Exceptional

Academic honesty

Academic honesty means that your work is based on your own original ideas and not copied from other people. However, you may draw on the work and ideas of others, but this must be acknowledged. This would be put into a reference list at the end of your investigation, known as a bibliography. In addition, you should use your own language and expression.

Record-keeping

Throughout the investigation, keep a journal, either on paper or online. This journal will also help you to demonstrate academic honesty. The journal will be of great assistance in focusing your efforts when writing your CBA 1 investigation.

- Make notes of any websites or books you use.
- You are encouraged to use a variety of support materials and present your work in a variety of formats.
- Keep a record of your actions so you can show your teacher how much time you are spending on your investigation.
- Remember to follow your teacher's advice and meet your CBA 1 timetable.
- The teacher is there to facilitate you, so do not be afraid to ask for guidance. The more focused your questions are, the better guidance your teacher can give you.

Evidence of learning

The following evidence is required

- A report
- Student research records

You must report your research and findings in a format of your choice. The report can be completed at the end of the investigation. If a typed or hand-written report is the format of choice, the total length of the report would typically be in the 400–600 words range (excluding tables, graphs, reference list and research records), but this should not be regarded as a rigid requirement.