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LESS STRESS MORE SUCCESS

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Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The Exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC Exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

1 Coordinate Geometry of the Line

aims

- ☐ To know where to find the coordinate geometry formulae in the booklet of formulae and tables
- ☐ To learn how to apply these formulae to procedural and in-context examination questions
- ☐ To gain the ability, with practice, to recall relevant techniques and tactics for the exam

Example

When geese fly in formation, they form an inverted V-shape.

(i) If the lines of geese can be represented by the equations $x + 2y - 10 = 0$ and $3x - 2y - 6 = 0$, find the coordinates of the leading goose.

(ii) After 1 hour, the leading goose has flown to a point $(37, 67)$.
Assuming the geese flew in a straight line and taking each unit to represent 1 km, find the distance travelled by the geese to the nearest km.

(iii) Hence, find the average flying speed in m/s.

Solution

(i)
$$\begin{array}{r} x + 2y - 10 = 0 \\ \underline{3x - 2y - 6 = 0} \\ 4x - 16 = 0 \\ 4x = 16 \\ x = 4 \end{array}$$

Sub $x = 4$ into $x + 2y - 10 = 0$

To get
$$\begin{aligned} 4 + 2y - 10 &= 0 \\ 2y - 6 &= 0 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

Leading goose position is $(4, 3)$.

(ii) Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (see booklet of formulae and tables page 18)

$(x_1, y_1) = (4, 3)$

$(x_2, y_2) = (37, 67)$

$$\text{Distance} = \sqrt{(37-4)^2 + (67-3)^2}$$
$$= \sqrt{1{,}089 + 4{,}096}$$
$$= \sqrt{5{,}185}$$
$$= 72{\cdot}00694411$$

Distance to the nearest km = 72 km

(iii) $\text{Speed} = \dfrac{\text{Distance}}{\text{Time}} = \dfrac{72 \times 1{,}000}{60 \times 60} = 20 \text{ m/sec}$

key point

This example is a revision of material from the Junior Certificate.

Example

The table shows temperatures in Celsius and the equivalent Fahrenheit.

Celsius (C)	50	65	80	95	100	120
Fahrenheit (F)	122	149	176	203	212	248

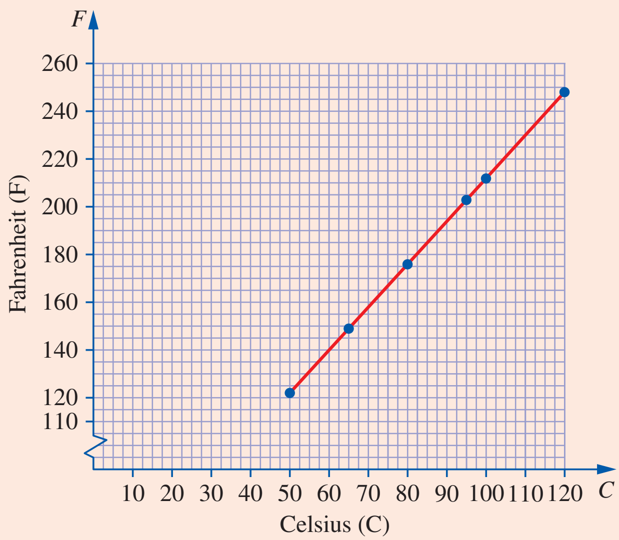
(i) Using a graph, investigate whether these values form a linear relationship.

(ii) Find this relationship in the form $F = aC + b$, where $a, b \in \mathbb{R}$ and C, F represent the temperature in Celsius and Fahrenheit, respectively.

(iii) Use this relationship to find the equivalent Fahrenheit temperature for $-30°$C.

Solution

(i)



The graph seems linear

(ii) To find the equation of the line, we need its slope and one point on it.

To find the slope, we use the two extreme points: (50, 122) and (120, 248).

Slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (see booklet of formulae and tables page 18)

$$= \frac{248 - 122}{120 - 50} = \frac{126}{70} = \frac{9}{5} \text{ or } 1{\cdot}8$$

Equation of a line: $y - y_1 = m(x - x_1)$ (see booklet of formulae and tables page 18)

$$F - 122 = \frac{9}{5}(C - 50)$$

$$F - 122 = \frac{9}{5}C - 90$$

$$F = \frac{9}{5}C - 90 + 122$$

$$F = \frac{9}{5}C + 32$$

(iii) $F = \dfrac{9}{5}(-30) + 32 = -54 + 32 = -22$

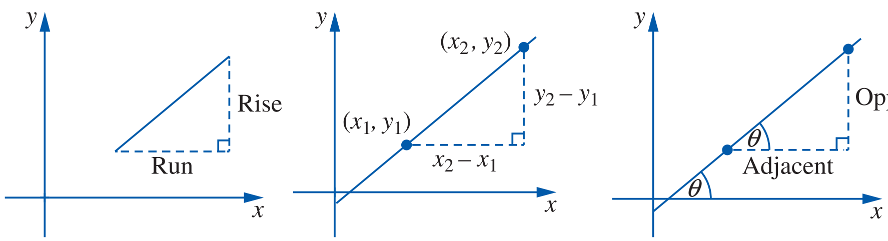
Thus $-30°\text{C} = -22°\text{F}$

Slope of a line

Slope of a line, m, given two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

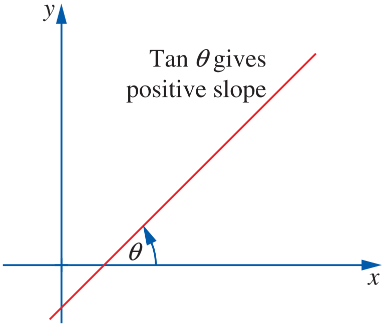
Slope is $\dfrac{\text{rise}}{\text{run}} = \tan\theta$, where θ is the angle the line makes with the positive sense of the x-axis.

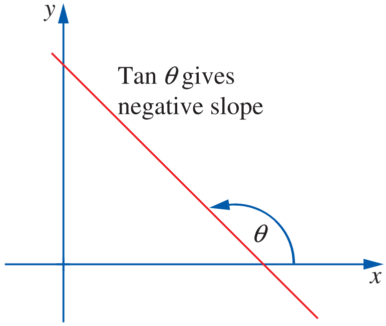


We say θ, the angle of inclination, is the angle formed between a line and the positive side of the x-axis.

The angle of inclination is always between 0° and 180°.

- It is always measured anticlockwise from the positive side of the x-axis.
- The slope m of any line is equal to the tangent of its angle of inclination: then $m = \tan\theta$ (where θ = angle of inclination).





Lines parallel to the axes

$y = 1$ is a line parallel to the x-axis through the point $(0, 1)$.

$x = 2$ is a line parallel to the y-axis through the point $(2, 0)$.

key point

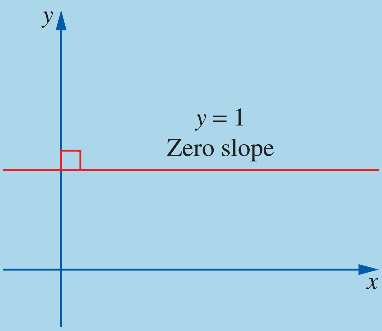
$y = 0$ is the equation of the x-axis.
$x = 0$ is the equation of the y-axis.

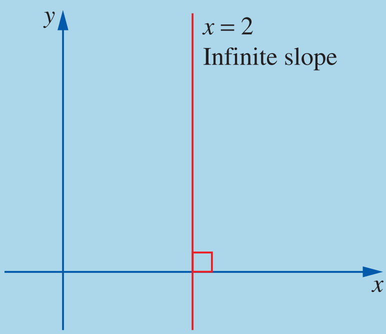
All horizontal lines have an angle of inclination of 0°.

Their slopes are zero.

All vertical lines have an angle of inclination of 90°.

Their slopes are infinitely steep.





Transformations of the plane

(a) Translation: A translation moves a point in a straight line.
(b) Central symmetry: Central symmetry is a reflection in a point.
(c) Axial symmetry: Axial symmetry is a reflection in a line.
(d) Axial symmetry in the axes or central symmetry in the origin.

key point

The following three patterns emerge and it is worth memorising them:

1. Axial symmetry in the x-axis $\rightarrow$ change the sign of y
2. Axial symmetry in the y-axis $\rightarrow$ change the sign of x
3. Central symmetry in the origin, $(0, 0) \rightarrow$ change the sign of both x and y

Note: Under a translation or a central symmetry, a line is mapped onto a parallel line.

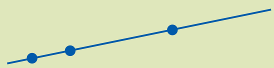
Example

Investigate whether $A(-4, 3)$, $B(-1, 6)$ and $C(7, 10)$ are collinear.

key point

Collinear: If three or more points lie on the same line, then the points are said to be collinear.

Note: Two points always lie on a line.



Solution

Method 1

The three points either form a straight line or they do not.
To decide which, we will find the slopes of AB and BC.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 3}{-1 - (-4)} = \frac{3}{3} = 1$$

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 6}{7 - (-1)} = \frac{4}{8} = \frac{1}{2}$$

As $m_{AB} \neq m_{BC}$, the points A, B and C are not collinear.
Note: We could have found the slope of m_{AC} as one of the two slopes.

Method 2

To find the area of $\triangle ABC$, use translation $(-4, 3) \rightarrow (0, 0)$ to get:

$A = (-4, 3) \rightarrow (0, 0)$ (Add 4 to x, subtract 3 from y)

$B = (-1, 6) \rightarrow (3, 3) = (x_1, y_1)$

$C = (7, 10) \rightarrow (11, 7) = (x_2, y_2)$

$$\therefore \text{ Area } \triangle ABC = \frac{1}{2}|x_1y_2 - x_2y_1| \quad \text{(see booklet of formulae and tables page 18)}$$

$$= \frac{1}{2}|(3)(7) - (11)(3)|$$

$$= \frac{1}{2}|21 - 33| = \frac{1}{2}|-12| = 6$$

Since the area of $\triangle ABC \neq 0$, the three points A, B and C do not form a straight line $\Rightarrow$ A, B and C are not collinear.

key point

If the area of $\triangle PQR = 0$, then we can state P, Q, R are collinear.

Division of a line segment in a given ratio

The coordinates of the point $C(x, y)$ which divides the line segment $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $a : b$ is given by:

Internal divisor

$$C(x, y) = \left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$$

(see booklet of formulae and tables page 18)

Example

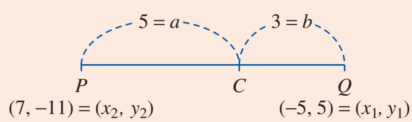
(i) $P(7, -11)$ and $Q(-5, 5)$ are two points. C is a point on $[PQ]$ such that $|PC| : |CQ| = 5 : 3$. Find the coordinates of C.

(ii) The point $R\left(-\frac{1}{2}, -1\right)$ divides the line segment $|VW|$ such that $|VR| : |RW| = 1 : 4$. If the coordinates of V are $(3, -3)$, find the coordinates of W.

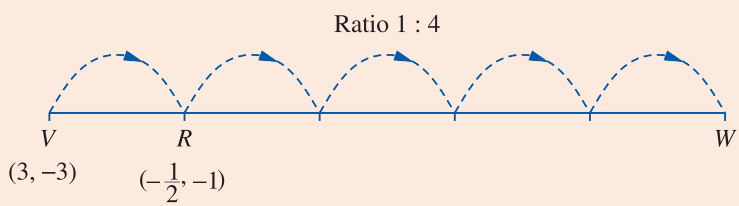
Solution

(i) $$C = \left(\frac{(5)(-5) + (3)(7)}{5 + 3}, \frac{(5)(5) + (3)(-11)}{5 + 3}\right)$$

$$C = \left(\frac{-25 + 21}{8}, \frac{25 - 33}{8}\right) = \left(-\frac{1}{2}, -1\right)$$



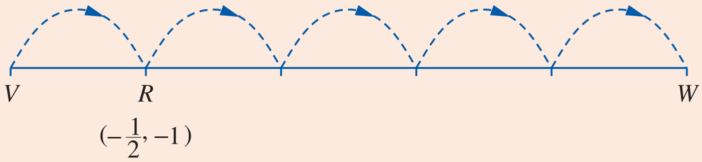
(ii) Using a translation



Notice translation $V \rightarrow R$ means $(3, -3) \rightarrow \left(-\frac{1}{2}, -1\right)$

x-component $-3\frac{1}{2}$

y-component $+2$



x-component $-\frac{1}{2} - 3\frac{1}{2} - 3\frac{1}{2} - 3\frac{1}{2} - 3\frac{1}{2} = -14\frac{1}{2}$

y-component $-1 + 2 + 2 + 2 + 2 = 7$

$$W = \left(-14\frac{1}{2}, 7\right)$$

exam Q

(i) The line $4x - 5y + k = 0$ cuts the x-axis at P and the y-axis at Q. Write down the coordinates of P and Q in terms of k.

(ii) The area of the triangle OPQ is 10 square units, where O is the origin. Find two possible values of k.

Solution

(i) P is on the x-axis $\Rightarrow y = 0$

$$4x - 5y + k = 0$$
$$4x - 5(0) + k = 0$$
$$x = -\frac{k}{4}$$

Thus, P has coordinates $\left(-\frac{k}{4}, 0\right)$.

Q is on the y-axis $\Rightarrow x = 0$

$$4x - 5y + k = 0$$
$$4(0) - 5y + k = 0$$
$$y = \frac{k}{5}$$

Thus, Q has coordinates $\left(0, \frac{k}{5}\right)$.

(ii) Points: $(0, 0)$, $\left(-\frac{k}{4}, 0\right)$, $\left(0, \frac{k}{5}\right)$ Given: area of $\triangle OPQ = 10$

$$\text{Area of } \triangle = \frac{1}{2}|x_1y_2 - x_2y_1| = 10$$

$$\frac{1}{2}\left|\left(-\frac{k}{4}\right)\left(\frac{k}{5}\right) - (0)(0)\right| = 10 \quad \text{(multiply by 2)}$$

$$\left|-\frac{k^2}{20}\right| = 20$$

$$-\frac{k^2}{20} = 20 \text{ or } -\frac{k^2}{20} = -20$$

$$k^2 = -400 \text{ or } k^2 = 400$$

Reject $\quad k = \pm 20$

key point

k^2 is always positive where $k \in R$, hence we reject $k^2 = -400$.

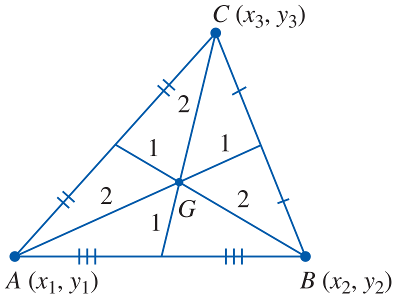
Concurrencies of a triangle

1. Centroid *G*

A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. The three medians of a triangle meet at a point called the centroid, *G*. *G* divides each median in the ratio 2 : 1.

Coordinates of $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

(see booklet of formulae and tables page 52)



exam focus

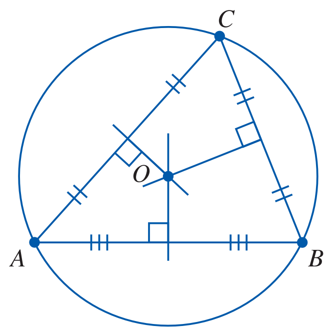
The following three geometry constructions using concurrencies may be easily incorporated into a coordinate geometry exam question. They are worth remembering.

key point

The centre of gravity of a triangular lamina is at its centroid.

2. Circumcentre *O*

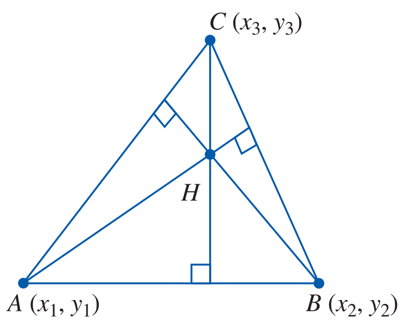
The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides.



3. Orthocentre *H*

An altitude of a triangle is a perpendicular line from a vertex to its opposite side. The orthocentre is the point of intersection of the altitudes.

Note: The centroid, circumcentre and orthocentre in a triangle all lie on a straight line called **Euler's line.**



Example

$P(7, 3)$ and $Q(-1, -5)$ are two points. R is the midpoint of $[PQ]$. Find the values of t if the line containing the point R and $S(t^2, t)$ is perpendicular to PQ.

Solution

We have two methods to find R.

Method 1: Average

$$\begin{array}{rl} P(7, 3) & \\ \underline{Q(-1, -5)} & \\ (6, -2) & \text{(add)} \\ R(3, -1) & \text{(divide by 2)} \end{array}$$

Method 2: Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{7 - 1}{2}, \frac{3 - 5}{2}\right)$$
$$R = (3, -1)$$

We have an equation in disguise, based on slopes, to find the values of t.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$PQ: \quad m_1 = \frac{-5 - 3}{-1 - 7}$$
$$= \frac{-8}{-8}$$
$$m_1 = 1$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$RS: \quad m_2 = \frac{t - (-1)}{t^2 - 3}$$
$$m_2 = \frac{t + 1}{t^2 - 3}$$

$$PQ \perp RS$$
$$m_1 \times m_2 = -1$$
$$(1)\left(\frac{t + 1}{t^2 - 3}\right) = -1$$
$$\frac{t + 1}{t^2 - 3} = -1$$
$$t + 1 = -1(t^2 - 3)$$
$$t + 1 = -t^2 + 3$$
$$t^2 + t - 2 = 0$$
$$(t + 2)(t - 1) = 0$$
$$t = -2 \text{ or } t = 1$$

key point

To prove whether or not two lines are perpendicular, do the following.

1. Find the slope of each line.
2. Multiply both slopes.
3. (i) If the answer in step 2 is -1, the lines are perpendicular.
   (ii) If the answer in step 2 is not -1, the lines are not perpendicular.

Note:

$bx - ay + k = 0$ is a line perpendicular to the line $ax + by + c = 0$.

key point

Parallel lines have equal slopes.
If $l_1 || l_2$, then $m_1 = m_2$.

Conversely, if two lines have equal slopes, then they are parallel.

Note:

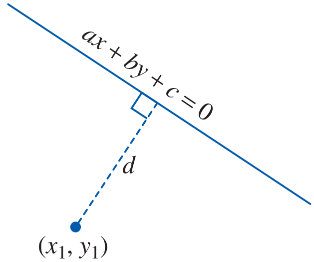
$ax + by + k = 0$ is a line parallel to the line $ax + by + c = 0$.

Perpendicular distance from a point to a line

The perpendicular distance, d, from the point (x_1, y_1) to the line $ax + by + c = 0$ is given by:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

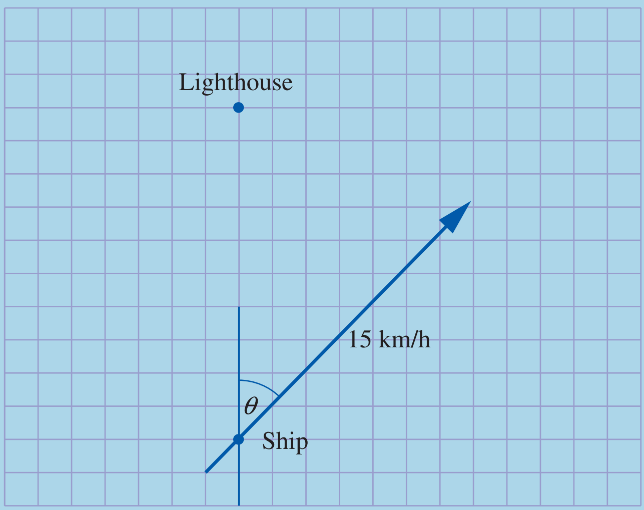
(see booklet of formulae and tables page 19)



exam Q

A ship is 10 km due south of a lighthouse at noon.

The ship is travelling at 15 km/h on a bearing of θ, as shown below, where $\theta = \tan^{-1}\frac{4}{3}$.



(i) On the diagram above, draw a set of coordinate axes that takes the lighthouse as the origin, the line east–west through the lighthouse as the x-axis and kilometres as units.

(ii) Find the equation of the line along which the ship is moving.

(iii) Find the shortest distance between the ship and the lighthouse during the journey.

(iv) At what time is the ship closest to the lighthouse?

exam focus

This question contains all the basic ingredients examiners like to use:

(a) Apply a simple diagram.

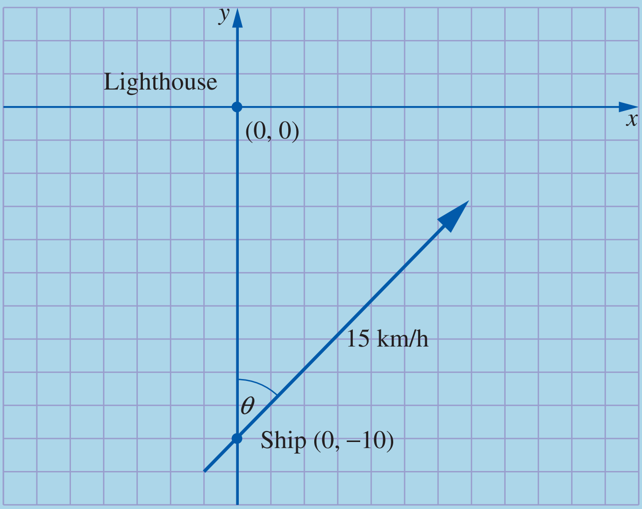
(b) Requires a standard formula but with a twist. In this case, $m \neq \tan\theta$, since θ is not the angle in the positive direction.

(c) Apply a more advanced formula.

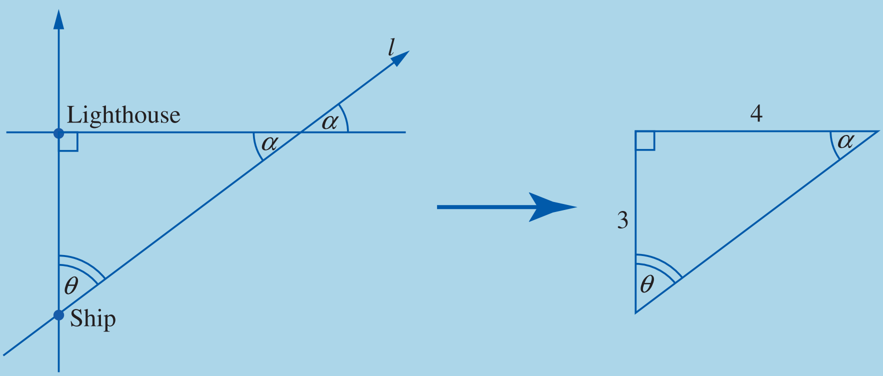
(d) To make the candidate think and use some techniques from other topics. In this instance, speed and time and Pythagoras' theorem.

Solution

(i)



(ii)



Given $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Then $\tan\theta = \dfrac{4}{3}$

Hence, $\tan \alpha = \frac{3}{4}$

The slope of line l is given by
$m = \tan \alpha = \frac{3}{4}$

The equation of l is given by
$y - y_1 = m(x - x_1)$

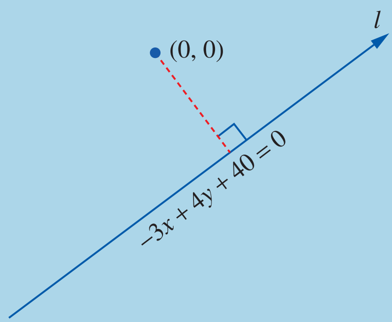
$(x_1, y_1) = (0, -10)$

$$y - (-10) = \frac{3}{4}(x - 0)$$
$$4y + 40 = 3x$$
$$4y - 3x + 40 = 0$$

key point

Given $\theta = \tan^{-1} k$
where $k \in \mathbb{R}$, then
$m = \tan \theta = \tan(\tan^{-1} k) = k$.

(iii) The shortest distance is given by the perpendicular distance.



$$\text{Perpendicular distance} = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$(x_1, y_1) = (0, 0)$

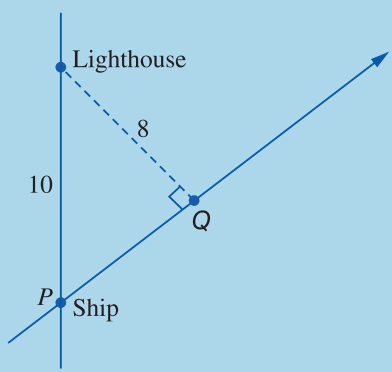
$a = -3$

$b = 4$

$c = 40$

$$\text{Shortest distance} = \frac{|(-3)(0) + (4)(0) + 40|}{\sqrt{(-3)^2 + (4)^2}}$$
$$= \frac{0 + 0 + 40}{\sqrt{9 + 16}} = 8 \text{ km}$$

(iv)



To find $|PQ|$, use the theorem of Pythagoras:

$(10)^2 = (8)^2 + |PQ|^2$

$100 = 64 + |PQ|^2$

$36 = |PQ|^2$

$6 \text{ km} = |PQ|$

Time required by the ship to travel 6 km given by:

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{6}{15} = 0{\cdot}4 \text{ hours}$$