LESS STRESS MORE SUCCESS

CONTENTS

Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The Exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC Exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

1 Coordinate Geometry of the Line

- To know where to find the coordinate geometry formulae in the booklet of formulae and tables
- To learn how to apply these formulae to procedural and in-context examination questions
- To gain the ability, with practise, to recall and select the appropriate technique required by the exam questions

Coordinating the plane and plotting points

Coordinates are used to describe the **position** of a point on a plane (flat surface).

Two lines are drawn at right angles to each other.

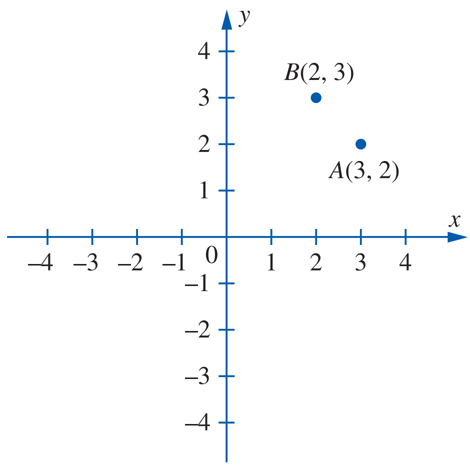
The horizontal line is called the ***x*-axis**.

The vertical line is called the ***y*-axis**.

The two axes meet at a point called the **origin**.

The plane is called the **Cartesian** (kar-tee-zi-an) plane.

Every point on the plane has two coordinates, an ***x*-coordinate** and a ***y*-coordinate**.



The coordinates are enclosed in brackets.

The x-coordinate is always written first, then a comma, followed by the y-coordinate.

On the diagram, the coordinates of the point A are (3, 2).

This is usually written as $A(3, 2)$.

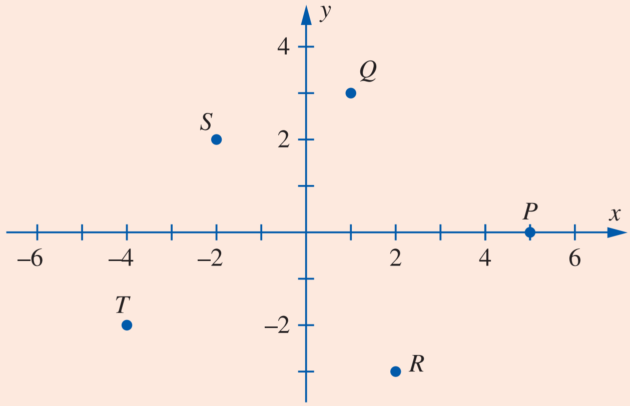
key point

In a couple (x, y) the order is important. The first number, x, is always **across**, **left** or **right**, and the second number, y, is always **up** or **down**.

The graph above shows the point $A(3, 2)$ is different to the point $B(2, 3)$.

Example

Write down the coordinates of the points P, Q, R, S and T



Solution

$P = (5, 0)$ $Q = (1, 3)$ $R = (2, -3)$ $S = (-2, 2)$ $T = (-4, -2)$

exam Q

An archaeologist has discovered various items at a site. The site is laid out in a grid and the position of each item is shown on the grid. The items found are a brooch (*B*), a plate (*P*), a ring (*R*), a statue (*S*) and a tile (*T*).

(a) Write down the co-ordinates of the position of each item.

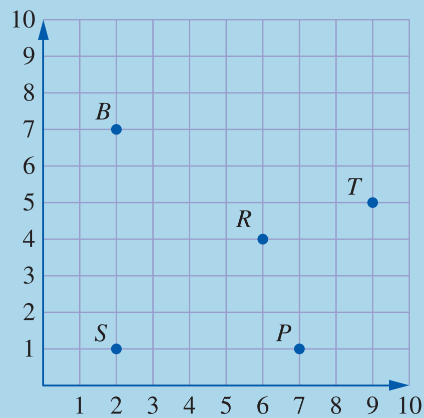
$B = (\ 2\ ,\ 7\ )$

$P = (\quad ,\quad)$

$R = (\quad ,\quad)$

$S = (\quad ,\quad)$

$T = (\quad ,\quad)$



(b) Each square of the grid represents 1 m^2. Find the total area of the grid.

(c) Which of the items is nearest to the tile (*T*)?

(d) Find the distance between the brooch (*B*) and the statue (*S*).

Solution

(a) $P = (7, 1)$ $R = (6, 4)$ $S = (2, 1)$ $T = (9, 5)$

(b) Count the grids: 10 up by 10 across
$= 10 \times 10 = 100\ m^2$

(c) By observation the ring (*R*) is nearest to the tile (*T*).

(d) $B(2, 7)$ to $S(2, 1) = 6$ m

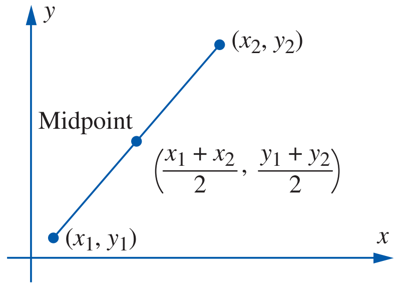
key point

Count each box as one unit.

Midpoint of a line segment

If (x_1, y_1) and (x_2, y_2) are two points, their midpoint is given by the formula:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



(See booklet of formulae and tables, page 18)

key point

When using coordinate geometry formulae, always allocate one point to be (x_1, y_1) and the other to be (x_2, y_2) before you use the formula.

Example

Noah is positioned at (8, 5) and a bus stop is positioned at (−10, 11). There is a traffic light exactly half way between Noah and the bus stop. Find the coordinates of the traffic light.

Solution

Midpoint (halfway) formula $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Let $(x_1, y_1) = (8, 5)$ and $(x_2, y_2) = (-10, 11)$

Coordinates of the traffic light $= \left(\frac{8 - 10}{2}, \frac{5 + 11}{2}\right) = \left(\frac{-2}{2}, \frac{16}{2}\right) = (-1, 8)$

exam Q

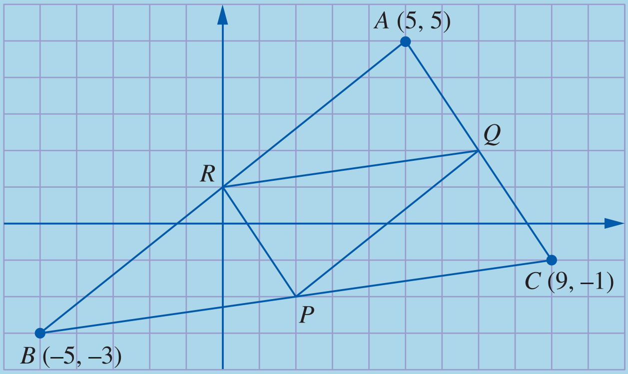
P, *Q* and *R* are the midpoints of the sides of the triangle *ABC*.

(i) Find the coordinates of *P*, *Q* and *R*.

(ii) The number of parallelograms in the diagram is

(a) 0 ☐
(b) 1 ☐
(c) 2 ☐
(d) 3 ☐

Tick the correct answer.

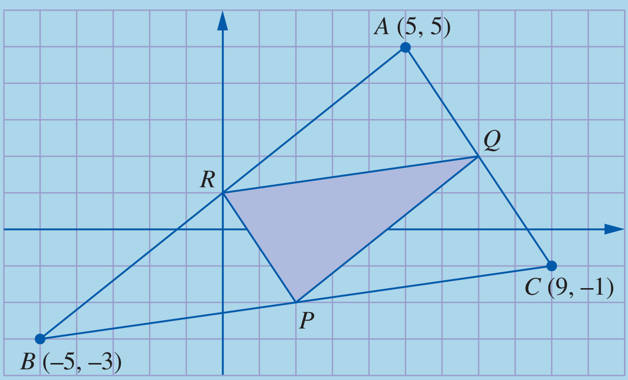


Solution

(i) Use the midpoint formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ three times.

Midpoint of [AB]	Midpoint of [AC]	Midpoint of [BC]
$(x_1, y_1) = (5, 5)$	$(x_1, y_1) = (5, 5)$	$(x_1, y_1) = (-5, -3)$
$(x_2, y_2) = (-5, -3)$	$(x_2, y_2) = (9, -1)$	$(x_2, y_2) = (9, -1)$
$R = \left(\frac{5-5}{2}, \frac{5-3}{2}\right)$	$Q = \left(\frac{5+9}{2}, \frac{5-1}{2}\right)$	$P = \left(\frac{-5+9}{2}, \frac{-3-1}{2}\right)$
$R = \left(\frac{0}{2}, \frac{2}{2}\right)$	$Q = \left(\frac{14}{2}, \frac{4}{2}\right)$	$P = \left(\frac{4}{2}, \frac{-4}{2}\right)$
$R = (0, 1)$	$Q = (7, 2)$	$P = (2, -2)$

(ii) **(d)** 3 ☑ The shaded triangle in the diagram forms half of three different parallelograms.



Translations

In mathematics, movement in a straight line is called a **translation**.

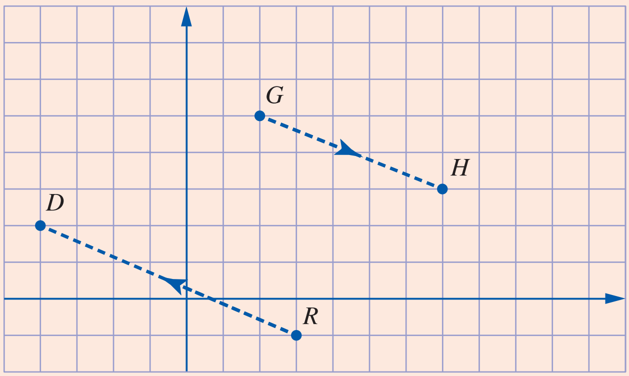
Under a translation, every point is moved the same distance in the same direction. A translation is one of several types of transformations on our course. See Chapter 4 for more on transformations.

Example

Describe the translation that maps the points:

(i) G to H

(ii) R to D



Solution

(i) $G \rightarrow H$ is described by 5 units to the right and 2 units down.

This translation can be written as $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

(ii) $R \rightarrow D$ is described by 7 units to the left and 3 units up.

This translation can be written as $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$.

Example

$A(-1, 1)$ and $B(4, -2)$ are two points.

Find the image of the point $(-1, 3)$ under the translation $\vec{AB}$

Solution

Under the translation $\vec{AB}$: $(-1, 1) \rightarrow (4, -2)$

Rule: Add 5 to x, subtract 3 from y, this can be written as $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

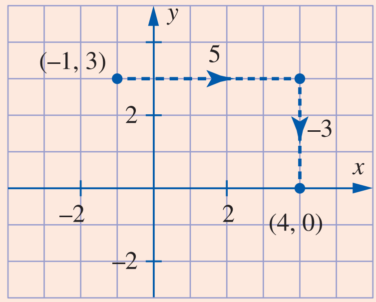
Method 1
Mathematical Method
(Apply the rule directly)

$(-1, 3) \rightarrow (-1 + 5, 3 - 3) = (4, 0)$

We say the image of $(-1, 3)$ is $(4, 0)$.

Method 2
Graphical Method

Plot the point $(-1, 3)$ and split the move into two parts:



- Horizontal move: 5 units to the right (add 5 to x)
- Vertical move: 3 units down (subtract 3 from y)

The image of $(-1, 3)$ is $(4, 0)$.

In some questions, we will be given the midpoint and one end point of a line segment and be asked to find the other end point.

To find the other end point use the following method:

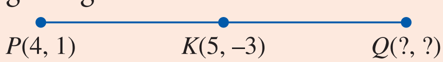
1. Draw a rough diagram.
2. Find the translation that maps (moves) the given end point to the midpoint.
3. Apply the same translation to the midpoint to find the other end point.

Example

If $K\,(5, -3)$ is the midpoint of $[PQ]$ and $P = (4, 1)$, find the coordinates of Q.

Solution

1. Rough diagram



2. Translation from P to K, $\overrightarrow{PK}$.

Rule: add 1 to x, subtract 4 from y. This can be written as $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

3. Apply this translation to K:

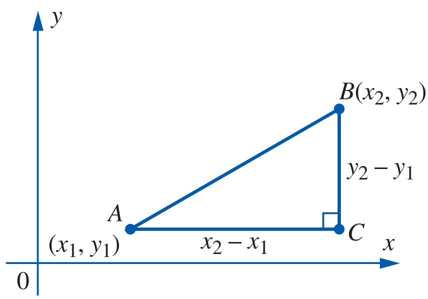
$K(5,-3) \rightarrow (5 + 1, -3 - 4) = (6,-7)$

$\therefore$ The coordinates of Q are $(6, -7)$.

Distance between two points

The given diagram shows the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

$|BC| = y_2 - y_1$ and $|AC| = x_2 - x_1$



Using the theorem of Pythagoras:

$$|AB|^2 = |AC|^2 + |BC|^2$$

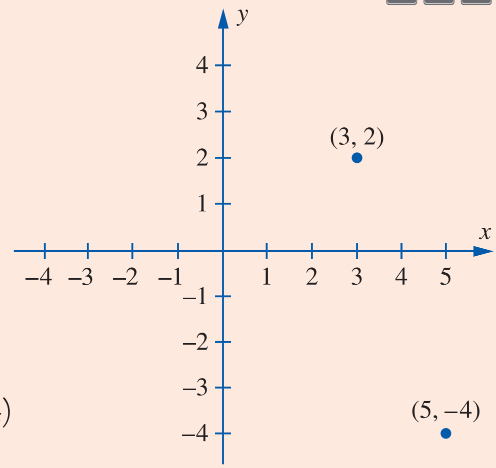
$$|AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (see booklet of formulae and tables page 18).

Example

Carol is positioned at the point (3, 2) and Siobhan is positioned at the point (5, −4). Find the distance between them.



Solution

Let $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (5, -4)$

Distance between Carol and Siobhan:

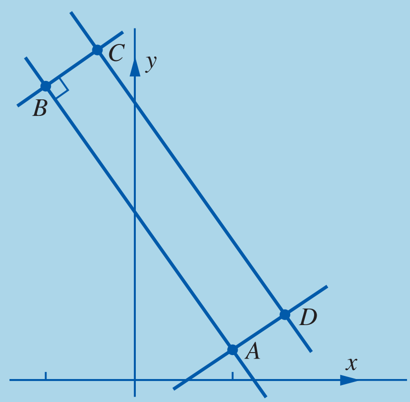
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5 - 3)^2 + (-4 - 2)^2}$$
$$= \sqrt{(2)^2 + (-6)^2}$$
$$= \sqrt{4 + 36}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10} \quad \text{(using a calculator)}$$

key point

At this stage, all numbers are always positive.

exam Q

ABCD is a rectangle with $A(3, 1)$ and $B(-3, 9)$. Given $|BC| = \frac{1}{5}|AB|$, calculate the area of ABCD.



Solution

Let $(x_1, y_1) = (3, 1)$ and $(x_2, y_2) = (-3, 9)$

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-3 - 3)^2 + (9 - 1)^2}$

$= \sqrt{(-6)^2 + (8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

$|BC| = \frac{1}{5}|AB| = \frac{1}{5}(10) = 2$

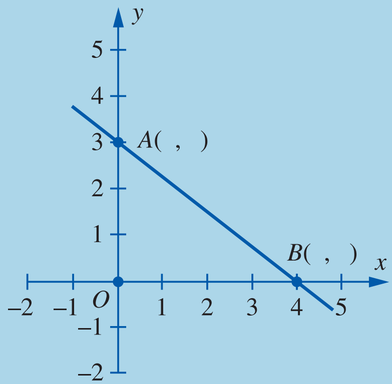
Area rectangle ABCD = (length)(width) = (10)(2) = 20 square units

(a) Write down the coordinates of the point A and the point B on the diagram.

(b) Use the distance formula to find $|AB|$.

(c) Write down the distance from O to A and the distance from O to B.

(d) Use the theorem of Pythagoras to find the length of the hypotenuse of the triangle OAB.



Solution

(a) $A = (0, 3)$

$B = (4, 0)$

(b) Let $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (4, 0)$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4 - 0)^2 = (0 - 3)^2}$$
$$= \sqrt{(4)^2 + (-3)^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

(c) $|OA| = 3$

$|OB| = 4$

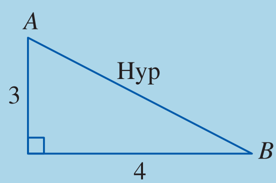
(d) The theorem of Pythagoras

$(\text{Hyp})^2 = (\text{Opp})^2 + (\text{Adj})^2$

$|AB|^2 = (3)^2 + (4)^2$

$|AB|^2 = 9 + 16 = 25$

$|AB| = 5$



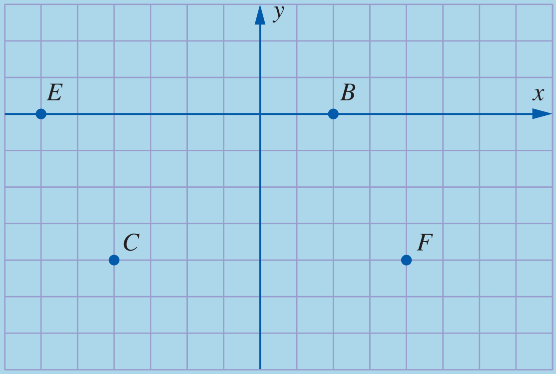
key point

Answer to part (b) = Answer to part (d).

exam Q

Given the points on the diagram:

B	C	E	F
(2, 0)	(−4, −4)	(−6, 0)	(4, −4)



(i) Find:

(a) |BE|
(b) |CF|
(c) |EC|
(d) |BF|

(ii) Hence, prove the triangle BCE is congruent to the triangle BCF.

Solution

(i) By observation from the diagram we can say:

(a) |BE| = 8

and **(b)** |CF| = 8

key point

For parts (a) and (b) count the width of each box as one unit.

We use $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for both |EC| and |BF|.

(c) |EC|

$(x_1, y_1) = (-6, 0)$

$(x_2, y_2) = (-4, -4)$

$$|EC| = \sqrt{(-4 - (-6))^2 + (-4 - 0)^2}$$
$$= \sqrt{(-4 + 6)^2 + (-4)^2}$$
$$= \sqrt{(2)^2 + 16}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$

(d) |BF|

$(x_1, y_1) = (2, 0)$

$(x_2, y_2) = (4, -4)$

$$|BF| = \sqrt{(4 - 2)^2 + (-4 - 0)^2}$$
$$= \sqrt{(2)^2 + (-4)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$

key point

|EC| = |BF| and |BE| = |CF|. These two pieces of information will be very useful in answering part (ii) of this question.

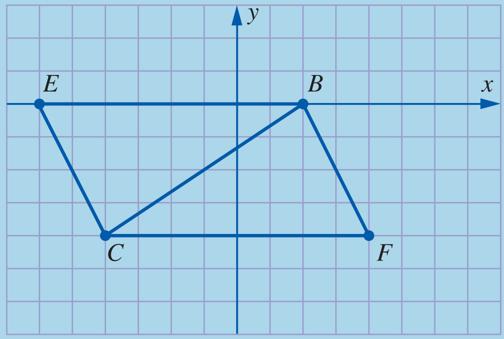
(ii) Consider ΔBCE and ΔBCF

|BC| = |BC| same

|EC| = |BF| = $\sqrt{20}$

|EB| = |CF| = 8

Hence (by SSS), ΔBCE is congruent (identical) to ΔBCF.



key point

The four cases for congruent triangles are covered in Chapter 2, Geometry.

exam focus

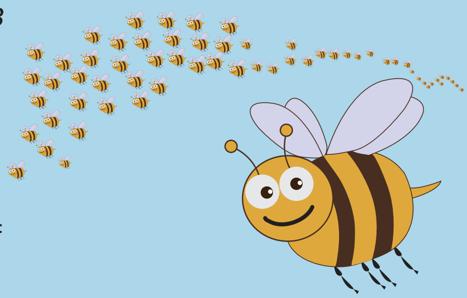
- Part (ii) above is an excellent example of an exam question linking two different topics on our course. In this case, we see coordinate geometry of the line linked with geometry theorems.
- In a recent exam, a similar question on congruence was asked, but it was worth very few marks. For not answering this part, candidates lost 1 mark out of a total of 27 marks awarded for the question.

Remember: Do not become disheartened, continue to do your best for every part of every question and you will do well.

exam Q

Henry travels in a swarm from Zone *A* to Zone *B*.

The swarms movement from Zone *A* to Zone *B* can be modelled by the translation that maps (0, 0) ⟶ (−1708, 503)



(i) If Henrys starting position in the swarm in Zone *A* is (1005, −98) find his position when the swarm move to Zone *B*.

(ii) Henrys best friend, Harriet, is also part of the swarm. Her position in Zone *B* is (−711, 399) find her starting position.

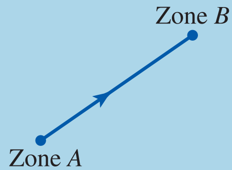
(iii) Find the distance in cm between Henry and Harriet as they travel in the swarm when each unit represents one millimetre.

Solution

(i) (0, 0) ⟶ (−1708, 503)

−1708 on the x-component

+503 on the y-component



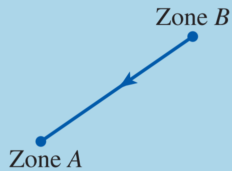
Henry: (1005, −98) ⟶ (1005 − 1708, −98 + 503) = (−703, 405)

(ii) Harriet: (?, ?) ⟶ (−711, 399)

Working backwards from Zone B to Zone A

+1708 on the x-component

−503 on the y-component



Harriet's Zone A position = (−711 + 1708, 399 − 503) = (997, −104)

(iii) The distance between Henry and Harriet in the swarm is always the same. Hence, we can find the distance from:

Zone A. (1005, −98) to (997, −104) or **Zone B.** (−703, 405) to (−711, 399)

Here, we find the distance from (1005, −98) to (997, −104).

Let $(x_1, y_1) = (1005, -98)$ and $(x_2, y_2) = (997, -104)$

$$\begin{aligned} \text{Distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(997 - 1005)^2 + (-104 - (-98))^2} \\ &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10 \text{ units} \end{aligned}$$

Since each unit is 1 mm,

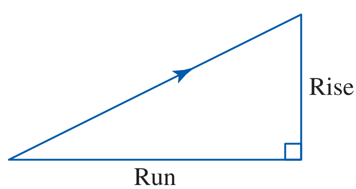
then 10 units is 10 mm = 1 cm

key point

As an exercise, you could verify the distance from (−711, 399) to (−703, 405) is also 10 units = 1 cm.

Slope of a line

All mathematical graphs are read from **left to right**.
The measure of the steepness of a line is called the **slope**.
The vertical distance, up or down, is called the **rise**.
The horizontal distance across is called the **run**.
The slope of a line is defined as:



$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

key point

The rise can also be negative and in this case it is often called the **'fall'**. If the rise is zero, then the slope is also zero.