LESS STRESS MORE SUCCESS

CONTENTS

Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

1 Number Systems

aims

- ☐ To learn what the symbols ℕ, ℤ, ℚ, and ℝ represent
- ☐ To be familiar with prime numbers, factors and the fundamental theorem of arithmetic
- ☐ To be able to find LCM and HCF as required

Natural numbers ℕ

The positive whole numbers 1, 2, 3, 4, 5, 6, . . . are also called the counting numbers. The dots indicate that the numbers go on forever and have no end (infinite).

exam Q

Give two reasons why $-7{\cdot}3$ is not a natural number.

Solution

Reason 1. It is a negative number

Reason 2. It is not a whole number (it is a decimal)

Factors (divisors)

key point

The factors of any whole number are the whole numbers that divide exactly into the given number, leaving no remainder.

1 is a factor of every number.

Every number is a factor of itself.

Example

Find the highest common factor of 18 and 45.

Solution

18		
1	×	18
2	×	9
3	×	6

45		
1	×	45
3	×	15
5	×	9

The common factors are 1, 3 and 9.

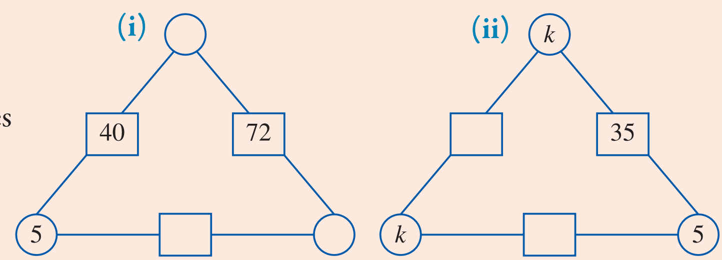
The highest common factor of 18 and 45 is 9.

key point

The highest common factor of two or more numbers is the largest factor that is common to each of the given numbers.

Example

In these productogons, the number in each square is the product of the numbers in the circles on each side of it. Find the missing numbers in each of these productogons.

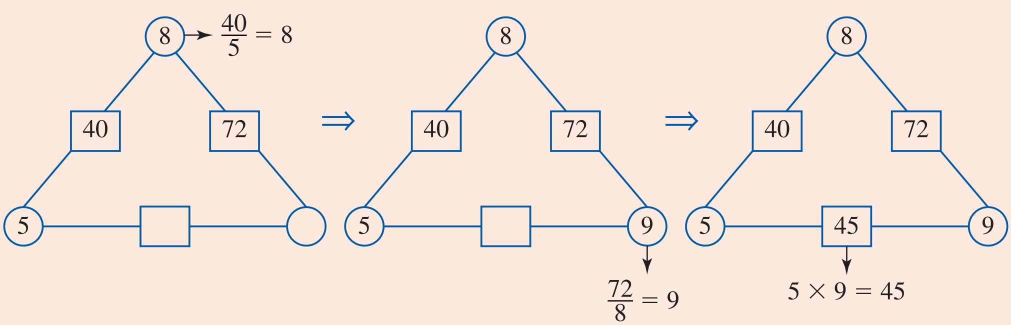


Solution

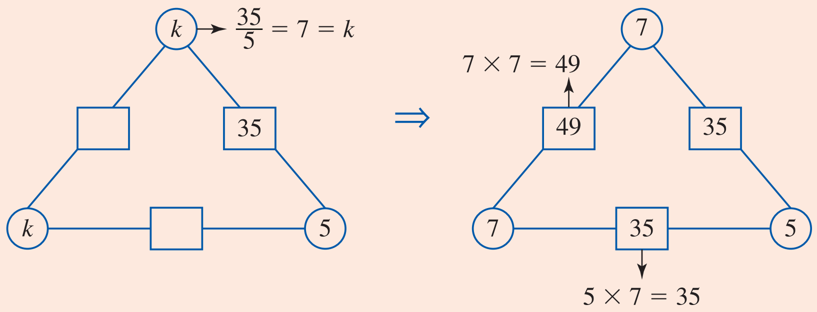
key point

The use of the word productogon in the question indicates we use multiplication. This is because product means multiply.

(i)



(ii)



Prime numbers

key point

A prime number is a whole number greater than 1 that has only two factors, 1 and itself.

The first 12 prime numbers are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 37.

There is an infinite number of prime numbers.

Numbers that have more than two factors are called composite numbers, e.g. 20 has 1, 2, 4, 5, 10, 20 as factors.

exam focus

The fundamental theorem of arithmetic states that any whole number greater than 1 can be written as the product of its prime factors in a **unique** way.
This will underpin many exam questions on number theory.

Prime factors

Any number can be expressed as a product of prime numbers. To express the number 180 as a product of its prime numbers, first divide by the smallest prime number that will divide exactly into it.

The smallest prime number 2	:	2 | 180
The smallest prime 2 again	:	2 | 90
The smallest prime 3	:	3 | 45
The smallest prime 3 again	:	3 | 15
The smallest prime 5	:	5 | 5
		1

So 180 expressed as a product of primes is:

$$2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$$

Example 1

For security, a credit card is encrypted using prime factors. A huge number is assigned to each individual card and it can only be verified by its prime factor decomposition. Find the 10-digit natural number which is assigned to the following credit cards whose prime factor decomposition is

(i) $2^2 \times 11 \times 13 \times 17^2 \times 19^3$

(ii) $2^7 \times 3^2 \times 5^2 \times 7^3 \times 23 \times 31$

Solution

By calculator: (i) 1133847572

(ii) 7043299200

Example 2

Gepetto makes wooden puppets. He has three lengths of wood which he wants to cut into pieces, all of which must be the same length and be as long as possible. The lengths of the three pieces of wood are 315cm, 357cm and 252cm.

(i) Express each of the three lengths as a product of primes.

(ii) Hence, calculate what length each piece should be and how many pieces he will have.

Solution

(i)

$$\begin{array}{r|r} 3 & 315 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline 7 & 7 \\ \hline & 1 \end{array} \qquad \begin{array}{r|r} 3 & 357 \\ \hline 7 & 119 \\ \hline 17 & 17 \\ \hline & 1 \end{array} \qquad \begin{array}{r|r} 2 & 252 \\ \hline 2 & 126 \\ \hline 3 & 63 \\ \hline 3 & 21 \\ \hline 7 & 7 \\ \hline & 1 \end{array}$$

$3^2 \times 5 \times 7$ $\qquad$ $3 \times 7 \times 17$ $\qquad$ $2^2 \times 3^2 \times 7$

(ii) By observation of the three product of primes above, the highest common factor (HCF) is given by $3 \times 7 = 21$.

Hence, each piece of wood should be 21cm long.

The number of pieces is given

by $\dfrac{315}{21} + \dfrac{357}{21} + \dfrac{252}{21}$

$= 15 + 17 + 12$

$= 44$

key point

3×7 is common to all three lengths.

Multiples and the lowest common multiple (LCM)

> The multiples of a number are found by multiplying the number by 1, 2, 3 . . . and so on.

The multiples of 4 are: 4, 8, 12, 16, 20...............

The multiples of 7 are: 7, 14, 21, 28, 35.............

> The **lowest common multiple** of two or more numbers is the **smallest multiple** that is common to each of the numbers.

In other words, the lowest common multiple is the **smallest** number into which each of the numbers will divide exactly.

For example, the lowest common multiple of 2, 4 and 7 is 28, as 28 is the smallest number into which 2, 4 and 7 will all divide exactly.

The lowest common multiple of two or more numbers is found with the following steps:

1. Write down the multiples of each number.
2. The lowest common multiple is the smallest (first) multiple they have in common.

Example

K is the set of natural numbers from 1 to 25, inclusive.

(i) List the elements of K that are multiples of 3.

(ii) List the elements of K that are multiples of 5.

(iii) Write down the lowest common multiple of 3 and 5.

Solution

K = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}

(i) Multiples of 3 = {3, 6, 9, 12, 15, 18, 21, 24}

(ii) Multiples of 5 = {5, 10, 15, 20, 25}

(iii) Lowest common multiple (LCM) is 15.
That is the smallest number that both sets have in common.

Integers $\mathbb{Z}$

Negative numbers are numbers below zero. Positive and negative **whole** numbers, including zero, are called integers.
Integers can be represented on a number line:

Integers to the right of zero are called **positive integers**.

Integers to the left of zero are called **negative integers**.

Example

At midnight on Christmas Eve the temperatures in some cities were as shown in the table.

(i) Which city recorded the
  (a) Lowest temperature
  (b) Highest temperature?

(ii) List the temperatures from coldest to hottest.

(iii) Which cities had a temperature difference of 6°C?

(iv) What is the difference in temperature between
  (a) Dublin and Moscow
  (b) Cairo and Dublin?

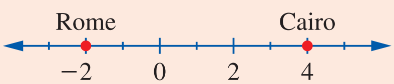
New York	2°C
Rome	−2°C
Dublin	−1°C
Moscow	−20°C
Cairo	4°C

Solution

(i) (a) Lowest temperature, −20°, in Moscow.
  (b) Highest temperature, 4°, in Cairo.

(ii) −20, −2, −1, 2, 4

(iii)



Rome and Cairo have a difference of 6°C

(iv) (a) Dublin and Moscow $= -1 - (-20) = -1 + 20 = 19°$
  (b) Cairo and Dublin $= 4 - (-1) = 4 + 1 = 5°$

Fractions (rational numbers)

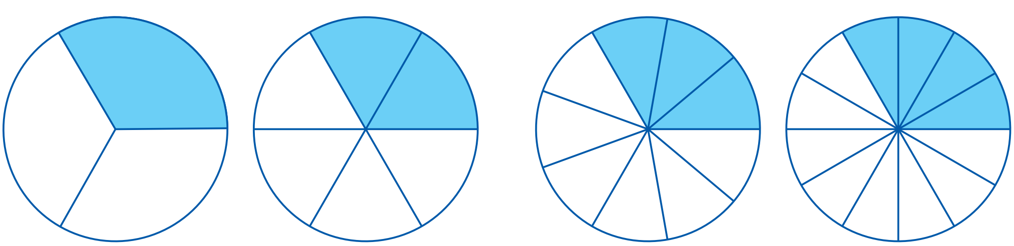
A fraction is written as two whole numbers, one over the other, separated by a bar. Equivalent fractions are fractions that are equal.
For example:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$$

This can be shown on a diagram where the same proportion is shaded in each circle.

key point

$$\text{Fraction} = \frac{\text{Numerator}}{\text{Denominator}}$$



$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$

A rational number (fraction) is a number that can be written as a ratio, $\frac{p}{q}$, of two integers, p and q, but $q \neq 0$.

Examples are $\frac{7}{2}$, $-\frac{11}{19}$, $8 = \frac{8}{1}$, $0 = \frac{0}{1}$, $5{\cdot}23 = \frac{523}{100}$

Rational numbers are denoted by the letter $\mathbb{Q}$.

Example

Bren is trying to subtract $\frac{1}{5}$ from $\frac{7}{8}$.

His attempt is shown here: $\frac{7}{8} - \frac{1}{5} = \frac{6}{3} = 2$.

(i) Explain what Bren has done wrong.

(ii) Write out the correct solution.

Solution

(i) It seems that Bren has subtracted the top numbers and subtracted the bottom numbers.

Key point: For subtraction or addition of fractions, we must find a common denominator.

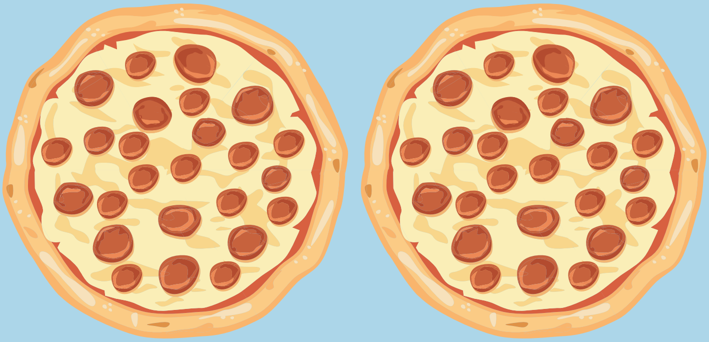
(ii) $\frac{7}{8} - \frac{1}{5} \Rightarrow$ common denominator $= 8 \times 5 = 40$

Then $\frac{7}{8} - \frac{1}{5} = \frac{(5)(7) - (8)(1)}{40} = \frac{35 - 8}{40} = \frac{27}{40}$.

Key point: See the chapter on algebraic expressions for more on addition/subtraction of fractions.

exam Q

Sheila orders two pizzas to divide evenly between herself and five friends.



(i) What fraction of a pizza will each person get? Write your fraction in its simplest form.

(ii) One of the friends gets a text and leaves before the pizza is delivered. What fraction will each person now get if the pizzas are divided evenly between those remaining?

(iii) Find how much extra pizza each person gets.

Solution

(i) 2 pizzas $\div$ 6 $= \frac{2}{6} = \frac{1}{3}$ each

(ii) 2 pizzas $\div$ 5 $= \frac{2}{5}$ each

(iii) Extra pizza $= \frac{2}{5} - \frac{1}{3}$

$= \frac{(2)(3) - (1)(5)}{15}$ (common denominator $= 5 \times 3 = 15$)

$= \frac{6 - 5}{15}$

$= \frac{1}{15}$

exam Q

Three students completed a test but got their results in different ways. The teacher told Karen that she got 0·7 of the questions correct. John was told he got 80% of the questions correct. David was told he got $\frac{3}{4}$ of the questions correct.

(i) Which student got the best result? Give a reason for your answer.

(ii) There were 20 questions on the test. How many questions each did Karen, John and David answer correctly?

(iii) Find the mean number of correct answers.

Solution

(i) Karen got $0{\cdot}7 = \dfrac{7}{10} = \dfrac{7 \times 100}{10}\% = 70\%$

John got 80%

David got $\dfrac{3}{4} = \dfrac{3 \times 100}{4}\% = 75\%$

By observation from the above work, John got the best result.

(ii) Karen got 70% of 20 $= \dfrac{70}{100} \times 20 = 14$ correct

John got 80% of 20 $= \dfrac{80}{100} \times 20 = 16$ correct

David got 75% of 20 $= \dfrac{75}{100} \times 20 = 15$ correct

(iii) Mean $= \dfrac{14 + 16 + 15}{3} = \dfrac{45}{3} = 15$

exam focus

The question was awarded 20 marks in total, as follows.

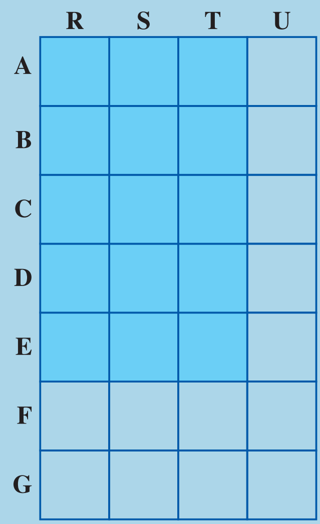
Part **i** 10 marks, with 5 marks awarded for one correct piece of work.

Part **ii** 5 marks, with 3 marks awarded for one correct piece of work.

Part **iii** 5 marks, with 3 marks awarded for one correct piece of work.

Hence, with 11 marks out of 20 marks awarded for no correct answers, you can see the importance of attempting every part of every question.

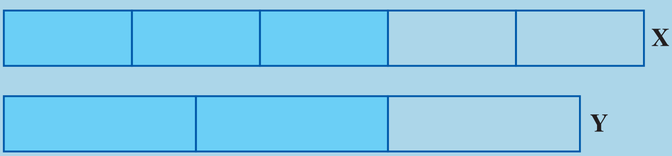
(i) (a) In the diagram below, what fraction of row **A** is shaded?



(b) In the same diagram, what fraction of column **R** is shaded?

(c) Using the diagram or otherwise, calculate the result when your fractions in part **(a)** and part **(b)** are multiplied.

(ii) Tim claims that the two fractions shown by the shading of the strips **X** and **Y** below are the same. Is Tim correct? Give a reason for your answer.



Solution

(i) (a) Count the shaded boxes in row A = 3

Answer $\frac{3}{4}$

(b) Count the shaded boxes in column R = 5

Answer $\frac{5}{7}$

(c) Method 1 $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$

Method 2 Count all the shaded boxes in the diagram = 15

Count every box in the diagram = 28

Answer $\frac{15}{28}$

(ii) Strip X has 3 boxes shaded out of 5 $= \frac{3}{5}$

Strip Y has 2 boxes shaded out of 3 $= \frac{2}{3}$

Since $\frac{3}{5} \neq \frac{2}{3}$, Tim is not correct.

exam Q

(i) Write $\frac{3}{8}$ as a decimal.

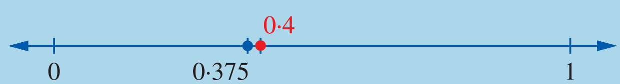
(ii) Represent the numbers $\frac{3}{8}$ and 0·4 on the number line below.

(iii) How could the number line in **(ii)** above help you decide which is the bigger of the two numbers?

Solution

(i) $\frac{3}{8} = 0{\cdot}375$

(ii)



(iii) The number to the right-hand side on a number line is always the bigger of the two numbers.

0·4 is the bigger number.