CONTENTS

Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The Exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC Exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

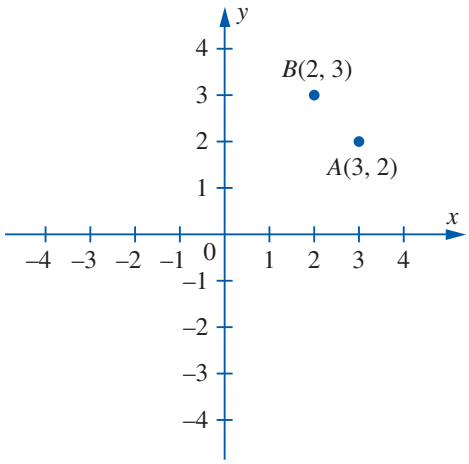
1 Coordinate Geometry of the Line

aims

- To know where to find the coordinate geometry formulae in the booklet of formulae and tables
- To learn how to apply these formulae to procedural and in-context examination questions
- To gain the ability, with practice, to recall and select the appropriate technique required by the exam questions

Coordinating the plane and plotting points

Coordinates are used to describe the **position** of a point on a plane (flat surface).
Two lines are drawn at right angles to each other.
The horizontal line is called the ***x*-axis**.
The vertical line is called the ***y*-axis**.
The two axes meet at a point called the **origin**.
The plane is called the **Cartesian** (kar-tee-zi-an) plane.
Every point on the plane has two coordinates, an ***x*-coordinate** and a ***y*-coordinate**.
The coordinates are enclosed in brackets.
The x-coordinate is always written first, then a comma, followed by the y-coordinate.
On the diagram, the coordinates of the point A are $(3, 2)$.
This is usually written as $A(3, 2)$.

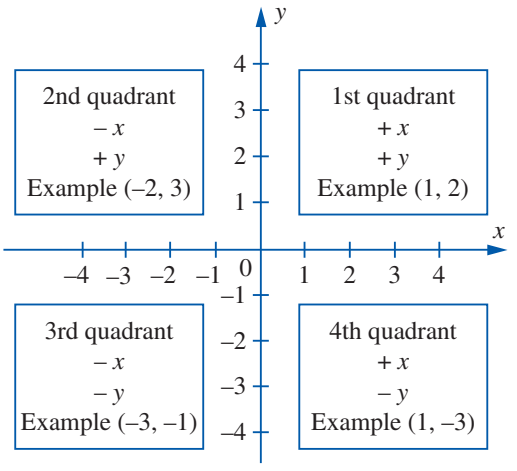


key point

- In a couple (x, y) the order is important
- The graph above shows that the point $A(3, 2)$ is different to the point $B(2, 3)$

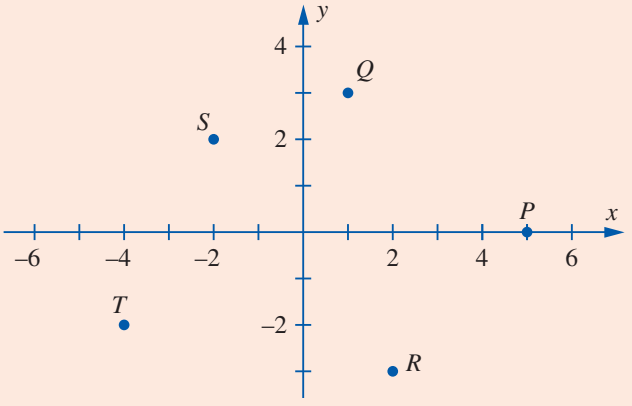
The four quadrants

The intersecting x-axis and y-axis divide the plane into four regions called **quadrants**. These are numbered 1st, 2nd, 3rd and 4th, as shown on the right.



Example

Write down the coordinates of the points P, Q, R, S and T.



Solution

$P = (5, 0)$ $Q = (1, 3)$ $R = (2, -3)$ $S = (-2, 2)$ $T = (-4, -2)$

Translation

In mathematics, movement in a straight line is called a **translation**.

Under a translation, every point is moved the same distance in the same direction.

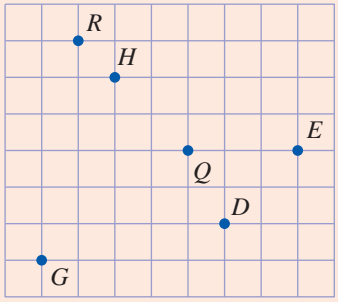
Example

Describe the translation that maps the points

(i) G to H

(ii) E to Q

(iii) R to D



Solution

(i) $G \rightarrow H$ is described by 2 units to the right and 5 units up.

This can be written as $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(ii) $E \rightarrow Q$ is described by 3 units to the left and 0 units up (or down). This can be written as $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

(iii) $R \rightarrow D$ is described by 4 units to the right and 5 units down. This can be written as $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

key point

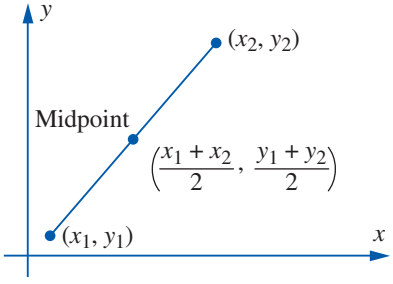
A more comprehensive treatment of translations can be found in Chapter 4 on transformation geometry.

Midpoint of a line segment

If (x_1, y_1) and (x_2, y_2) are two points, their midpoint is given by the formula:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

(See booklet of formulae and tables page 18)



key point

Before using coordinate geometry formulae, always allocate one point to be (x_1, y_1) and the other to be (x_2, y_2).

Example

$A(8, 5)$ and $B(-10, 11)$ are two points. Find the midpoint of $[AB]$.

Solution

$$\text{Midpoint formula} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Let $(x_1, y_1) = (8, 5)$ and $(x_2, y_2) = (-10, 11)$

$$\text{Midpoint} = \left(\frac{8 - 10}{2}, \frac{5 + 11}{2}\right) = \left(\frac{-2}{2}, \frac{16}{2}\right) = (-1, 8)$$

In some questions, we will be given the midpoint and one end point of a line segment. We will be asked to find the other end point.

To find the other end point, use the following method:

1. Draw a rough diagram.
2. Find the translation that maps (moves) the given end point to the midpoint.
3. Apply the same translation to the midpoint to find the other end point.

Example

If $K(5, -3)$ is the midpoint of $[PQ]$ and $P = (4, 1)$, find the coordinates of Q.

Solution

1. Rough diagram:

$P(4, 1)$ $\quad$ $K(5, -3)$ $\quad$ $Q(?, ?)$

2. Translation from P to K, $\overrightarrow{PK}$. Rule: add 1 to x, subtract 4 from y.

This can be written as $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

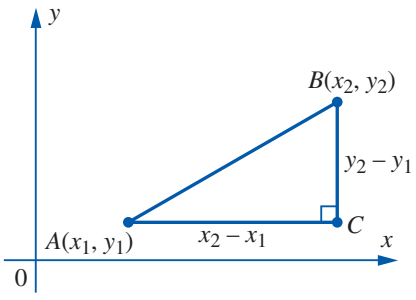
3. Apply this translation to K:

$$K(5, -3) \rightarrow (5 + 1, -3 - 4) = (6, -7)$$

$\therefore$ The coordinates of Q are $(6, -7)$.

Distance between two points

The given diagram shows the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



$|BC| = y_2 - y_1$ and $|AC| = x_2 - x_1$

Using the theorem of Pythagoras:

$$|AB|^2 = |AC|^2 + |BC|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\therefore \quad |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance (length) between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (*See* booklet of formulae and tables page 18).

Example

Find the distance between the points $A(3, 2)$ and $B(5, -4)$.

Solution

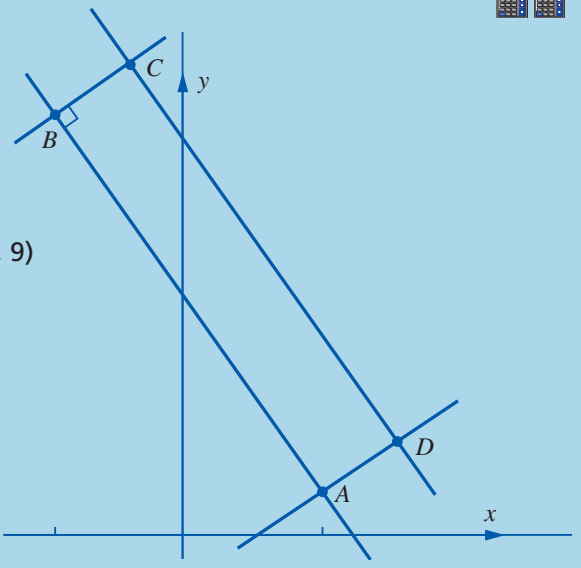
Let $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (5, -4)$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5 - 3)^2 + (-4 - 2)^2}$$
$$= \sqrt{2^2 + (-6)^2}$$
$$= \sqrt{4 + 36}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

At this stage the numbers are always positive.

exam Q

ABCD is a rectangle with $A(3, 1)$ and $B(-3, 9)$. Given $|BC| = \frac{1}{5}|AB|$, calculate the area of ABCD.



Solution:

Let $(x_1, y_1) = (3, 1)$ and $(x_2, y_2) = (-3, 9)$

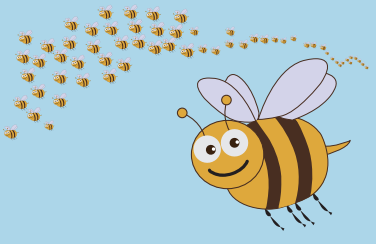
$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-3 - 3)^2 + (9 - 1)^2}$$
$$= \sqrt{(-6)^2 + (8)^2}$$
$$= \sqrt{36 + 64} = \sqrt{100} = 10$$

$$|BC| = \frac{1}{5}|AB| = \frac{1}{5}(10) = 2$$

Area rectangle ABCD = (length)(breadth) = (10)(2) = 20 square units

exam Q

Henry the bee travels in a swarm from Zone *A* to Zone *B*.



The swarm's movement from zone A to zone B can be modelled by the translation that maps $(0, 0) \rightarrow (-1708, 503)$.

(i) If Henry's starting position in the swarm in Zone *A* is $(1005, -98)$ find his position when the swarm moves to Zone *B*.

(ii) Henry's best friend, Harriet, is also part of the swarm. If her position in Zone *B* is $(-711, 399)$, find her starting position.

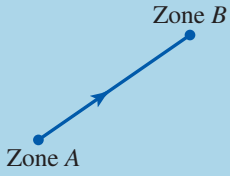
(iii) Find the distance in cm between Henry and Harriet as they travel in the swarm when each unit is one mm.

Solution

(i) (0, 0) → (−1708, 503)

−1708 on the *x*-component

+503 on the *y*-component



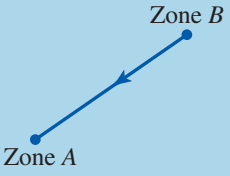
Henry: (1005, −98) → (1005 − 1708, −98 + 503) = (−703, 405)

(ii) Harriet: (?, ?) → (−711, 399)

Working backwards from Zone *B* to Zone *A*

+1708 on the *x*-component

−503 on the *y*-component.



Harriet's Zone *A* position = (−711 + 1708, 399 − 503) = (997, −104)

(iii) The distance between Henry and Harriet in the swarm is always the same. Hence, we can find the distance from:

Zone A. (1005, −98) to (997, −104) or **Zone B.** (−703, 405) to (−711, 399)

Here we find the distance from (1005, −98) to (997, −104).

Let $(x_1, y_1) = (1005, -98)$ and $(x_2, y_2) = (997, -104)$

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(997 - 1005)^2 + (-104 - (-98))^2}$$

$$= \sqrt{(-8)^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10 \text{ units}$$

Since each unit is 1 mm

Then 10 units is 10 mm = 1 cm

key point

As an exercise, you could verify that the distance from (−711, 399) to (−703, 405) is also 10 units = 1 cm.

exam Q

When geese fly in formation, they form an inverted v-shape.

(i) If the lines of geese can be represented by the equations $2x + y - 11 = 0$ and $3x - 2y - 6 = 0$, find the coordinates of the leading goose.

After 1 hour, the leading goose has flown to a point (37, 67).

(ii) Assuming the geese flew in a straight line and taking each unit to represent 1 km, find the distance travelled by the geese to the nearest km.

(iii) Hence, find the average flying speed in m/s.

Solution

(i) Solving the linear equations in two variables:

$$\begin{aligned} 2x + y &= 11 && ① \\ 3x - 2y &= 6 && ② \\ 4x + 2y &= 22 && ① \times 2 \\ 3x - 2y &= 6 && ② \\ 7x &= 28 && \text{(Add)} \\ x &= 4 \end{aligned}$$

Put $x = 4$ into ① or ②

$$2x + y = 11 \quad ①$$

$$2(4) + y = 11$$

$$8 + y = 11$$

$$y = 3$$

exam focus

Solving linear equations is a skill you must know. Another example appears later in this chapter.

$\therefore$ The solution is $x = 4$ and $y = 3$ or $(4, 3)$

(ii) Use distance formula $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Let $(x_1, y_1) = (4, 3)$ and $(x_2, y_2) = (37, 67)$

Distance $= \sqrt{(37 - 4)^2 + (67 - 3)^2} = \sqrt{1{,}089 + 4{,}096} = \sqrt{5{,}185}$

$= 72{\cdot}00694411$

Distance to nearest km $= 72$ km

(iii) Speed $= \dfrac{\text{Distance}}{\text{Time}} = \dfrac{72 \times 1{,}000}{60 \times 60} = 20$ m/sec

exam focus

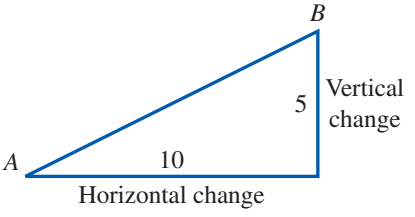
The exam may contain in-context questions at any stage. Be prepared to employ techniques learned elsewhere, as in the above question where Speed $= \dfrac{\text{Distance}}{\text{Time}}$. This would seem to have no link to coordinate geometry.

Slope of a line

The slope of the line AB is defined as the

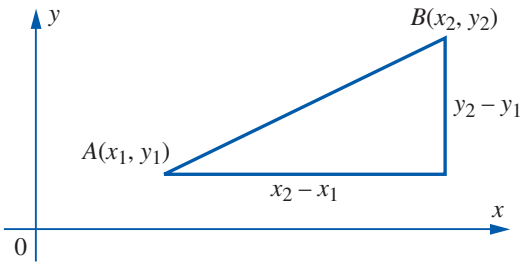
$$\frac{\text{vertical change}}{\text{horizontal change}} \quad \text{or} \quad \frac{\text{rise}}{\text{run}}$$

$$\text{The slope of } AB = \frac{5}{10} = \frac{1}{2}.$$



In the diagram on the right, the slope of AB is found by getting the

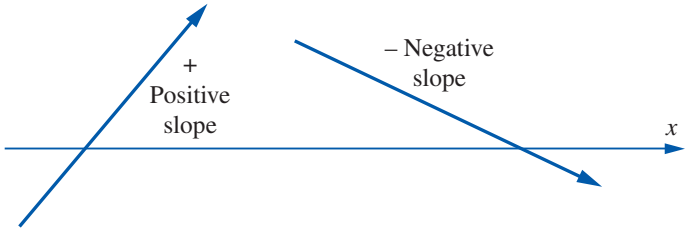
$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{y_2 - y_1}{x_2 - x_1}$$



Slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (See booklet of formulae and tables page 18)

Positive and negative slopes

As we move from left to right the slope is positive if the line is rising and the slope is negative if the line is falling



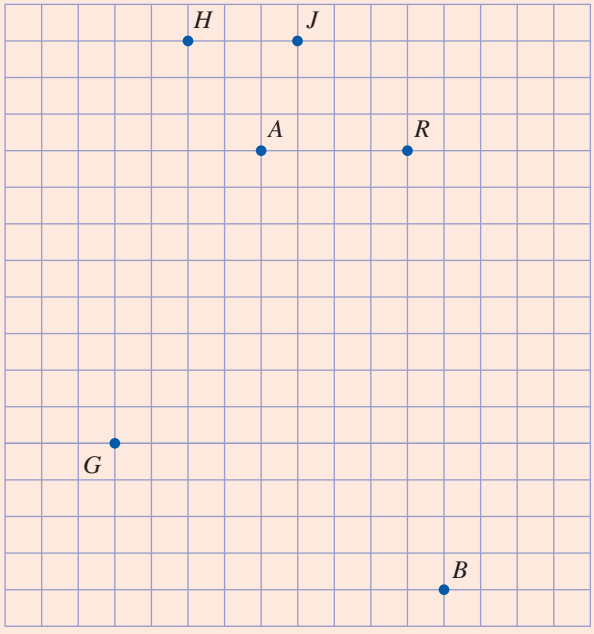
Example

Write down the slopes of the following lines in the diagram.

(i) GR **(ii)** BR

(iii) HJ **(iv)** GA

(v) AB **(vi)** BG



Solution

Use $\frac{\text{rise}}{\text{run}}$ (by counting the boxes) in each case to find

(i) Slope GR $= \frac{8}{8} = 1$ — Line going up $\Rightarrow$ positive slope

(ii) Slope BR $= -\frac{12}{1} = -12$ — Line going down $\Rightarrow$ negative slope

(iii) Slope HJ $= \frac{0}{3} = 0$ — Horizontal line $\Rightarrow$ slope-zero

(iv) Slope GA $= \frac{8}{4} = 2$ — Line going up $\Rightarrow$ positive slope

(v) Slope AB $= -\frac{12}{5}$ — Line going down $\Rightarrow$ negative slope

(vi) Slope BG $= -\frac{4}{9}$ — Line going down $\Rightarrow$ negative slope

exam Q

An accountant plots the straight line value of a computer over a three-year period on the given graph.

(i) Find the slope of the line.

(ii) Hence write down the average rate of change in the value of the computer.

Justify your answer.