CONTENTS

LESS STRESS MORE SUCCESS

Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

1 Number Systems

aims

- ☐ To learn what the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ and $\mathbb{R}$ represent.
- ☐ To be familiar with prime numbers, factors and the fundamental theorem of arithmetic.
- ☐ To be able to find LCM and HCF as required.
- ☐ To have a clear understanding of and be able to complete calculations using the order of operations.

Number sets

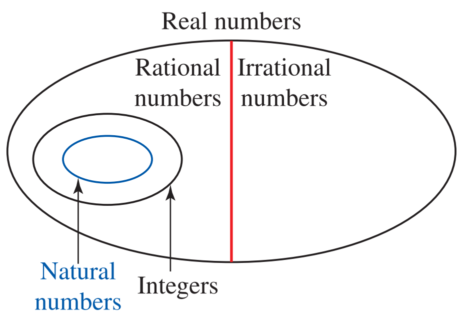
Four of the five number sets required on our course are to be found in the booklet of formulae and tables. It gives us

	Number sets
$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \cdots\}$	Natural numbers
$\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$	Integers
$\mathbb{Q} = \left\{ \frac{p}{q} \middle\| p \in \mathbb{Z}, \quad q \in \mathbb{Z}, \quad q \neq 0 \right\}$	Rational numbers
$\mathbb{R}$	Real numbers

The set not given in the tables is the set of irrational numbers, represented by $\mathbb{R}\backslash\mathbb{Q}$.

We meet the set of irrational numbers later in this chapter.

A Venn diagram of the number system looks like this:



Natural numbers $\mathbb{N}$

The positive whole numbers 1, 2, 3, 4, 5 . . . are also called the counting numbers. The dots indicate that the numbers go on forever and have no end (infinite).

exam Q

Give two reasons why $-7{\cdot}3$ is not a natural number.

Solution

Reason 1: It is a negative number.

Reason 2: It is not a whole number (it is a decimal).

Factors (divisors)

key point

The factors of any whole number are the whole numbers that divide exactly into the given number, leaving no remainder.

- 1 is a factor of every number.
- Every number is a factor of itself.

Example

Find the factors of 18.

Find the factors of 45.

Hence find the highest common factor of 18 and 45.

Solution

18	45
1×18	1×45
2×9	3×15
3×6	5×9

The common factors are 1, 3 and 9.

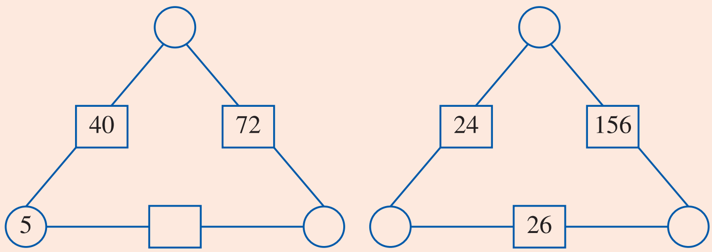
$\therefore$ The highest common factor of 18 and 45 is 9.

key point

The highest common factor (HCF) of two or more numbers is the largest factor that is common to each of the given numbers.

Example

In these productogons, the number in each square is the product of the numbers in the circles on each side of it. Find the missing numbers in each of these productogons.

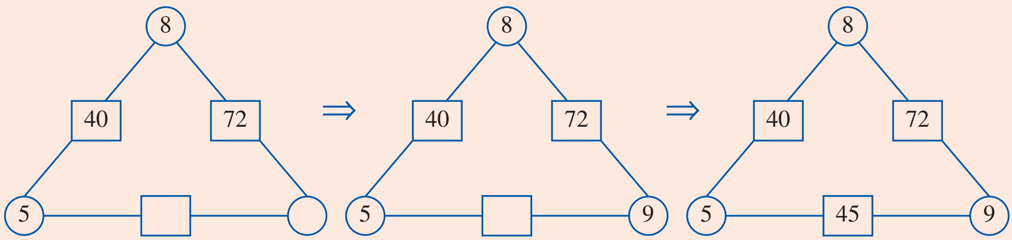


Solution

key point

The use of the word productogon in the question indicates we use multiplication. This is because product means multiply.

This first one is very straightforward.

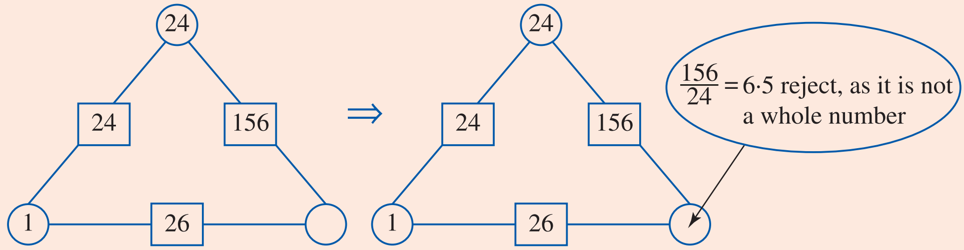


The second one is more challenging.
The method of trial and improvement (yes, guesswork!) is used here. Of the three given numbers, 24 and 26 would seem to have the easiest factors for us to find.

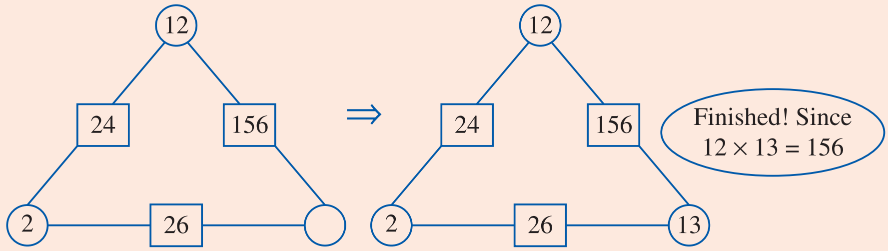
24	26
1×24	1×26
2×12	2×13
3×8	
4×6	

This indicates the bottom left-hand number is either 1 or 2, as they are the only factors common to both 24 and 26.

Let's put 1 into the bottom left-hand corner:



Now let's put 2 into the bottom left-hand corner:



Prime numbers

key point

A prime number is a whole number greater than 1 that has only two factors, 1 and itself.

The first 12 prime numbers are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 37.

There is an infinite number of prime numbers.

key point

Numbers that have more than two factors are called composite numbers.

The first 12 composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20 and 21.
There is an infinite number of composite numbers.

exam focus

The fundamental theorem of arithmetic states that any whole number greater than 1 can be written as the product of its prime factors in a **unique** way. This will underpin many exam questions on number theory.

Prime factors

Any number can be expressed as a product of prime numbers. To express the number 180 as a product of its prime numbers, first divide by the smallest prime number that will divide exactly into it.

The smallest prime number 2	:	2	180
The smallest prime 2 again	:	2	90
The smallest prime 3	:	3	45
The smallest prime 3 again	:	3	15
The smallest prime 5	:	5	5
			1

So 180 expressed as a product of primes is $2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$.

Example

For security, a credit card is encrypted using prime factors. A huge number is assigned to each individual card and it can only be verified by its prime factor decomposition. Find the 10-digit natural number which is assigned to the following credit cards whose prime factor decomposition is

(i) $2 \times 3 \times 11 \times 13 \times 17^2 \times 19^3$

(ii) $2^7 \times 3^2 \times 5^2 \times 7^3 \times 23 \times 31$

Solution

By calculator (i) 1700771358

(ii) 7043299200

Example

Geppetto makes wooden puppets. He has four lengths of wood which he wants to cut into pieces, all of which must be the same length and be as long as possible. The lengths of the four pieces of wood are 315cm, 357cm, 210cm and 252cm.

(i) Express each of the four lengths as a product of primes.

(ii) Hence, calculate what length each piece should be and how many pieces he will have.

Solution

(i)

$$\begin{array}{r|l} 3 & 315 \\ 3 & 105 \\ 5 & 35 \\ 7 & 7 \\ & 1 \end{array} \qquad \begin{array}{r|l} 3 & 357 \\ 7 & 119 \\ 17 & 17 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 210 \\ 3 & 105 \\ 5 & 35 \\ 7 & 7 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 252 \\ 2 & 126 \\ 3 & 63 \\ 3 & 21 \\ 7 & 7 \\ & 1 \end{array}$$

$3^2 \times 5 \times 7 \qquad 3 \times 7 \times 17 \qquad 2 \times 3 \times 5 \times 7 \qquad 2^2 \times 3^2 \times 7$

(ii) By observation of the four 'products of primes' above:

The highest common factor (HCF) is given by $3 \times 7 = 21$.

Hence, each piece of wood should be 21cm long.

The number of pieces is given by

$$\frac{315}{21} + \frac{357}{21} + \frac{210}{21} + \frac{252}{21}$$

$$= 15 + 17 + 10 + 12$$

$$= 54$$

key point

3×7 is common to all four lengths.

Integers $\mathbb{Z}$

Negative numbers are numbers below zero. Positive and negative **whole** numbers including 0 are called integers.

Integers can be represented on a number line:

Integers to the right of zero are called **positive integers**.

Integers to the left of zero are called **negative integers**.

Example

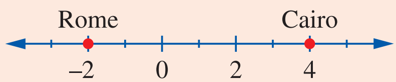
At midnight on Christmas Eve the temperatures in some cities were as shown in the table.

New York	2°C
Rome	−2°C
Dublin	−1°C
Moscow	−20°C
Cairo	4°C

(i) Which city recorded the
  (a) Lowest temperature
  (b) Highest temperature?
(ii) List the temperatures from coldest to hottest.
(iii) Which cities had a temperature difference of 6°C?
(iv) What is the difference in temperature between
  (a) Dublin and Moscow
  (b) Cairo and Dublin?

Solution

(i) (a) Lowest temperature, −20°, in Moscow
  (b) Highest temperature, 4°, in Cairo
(ii) −20, −2, −1, 2, 4
(iii)



Rome and Cairo had a difference of 6°C.

(iv) (a) Dublin and Moscow $= -1 - (-20) = -1 + 20 = 19°$
  (b) Cairo and Dublin $= 4 - (-1) = 4 + 1 = 5°$

Multiplication and division of two integers

The following two rules are applied to the multiplication or division of two integers.

1. If the signs are the same, then the answer will be positive.

e.g. $\dfrac{-10}{-2} = +5;\quad (-10)(-2) = +20;\quad \dfrac{+10}{+2} = +5$

2. If the signs are different, then the answer will be negative.

e.g. $\dfrac{-10}{+2} = -5;\quad (+10)(-2) = -20;\quad \dfrac{+10}{-2} = -5$

Example

Find the missing number in each box.

(i) $\square \times 5 = -10$ **(ii)** $8 \times \square = -24$

(iii) $-12 \div \square = 4$ **(iv)** $\square \div -9 = -4$

Solution

(i)
$$\square \times 5 = -10$$
$$\square = \frac{-10}{5}$$
$$\square = -2$$

(ii)
$$8 \times \square = -24$$
$$\square = \frac{-24}{8}$$
$$\square = -3$$

(iii)
$$-12 \div \square = 4$$
$$\frac{-12}{\square} = 4$$
$$-12 = 4\square$$
$$\frac{-12}{4} = \square$$
$$-3 = \square$$

(iv)
$$\square \div -9 = -4$$
$$\frac{\square}{-9} = -4$$
$$\square = (-4)(-9)$$
$$\square = 36$$

Order of operations

A memory aid for the order of operations is BEMDAS (brackets, exponents, multiplication and division, addition and subtraction).

key point

Exponent(s) = Power(s) = Index (indices)

Example

Calculate: **(i)** $8 + 108 \div -9$ **(ii)** $10 \times 4 - 30 \div 6 + 19$

Solution

(i) $8 + 108 \div -9$

$= 8 - 12$ Division

$= -4$ Subtraction

(ii) $10 \times 4 - 30 \div 6 + 19$

$= 40 - 30 \div 6 + 19$ Multiplication

$= 40 - 5 + 19$ Division

$= 59 - 5$ Addition

$= 54$ Subtraction

Example

Calculate $4(5 - 3)^2 + 24 \div (6 - 2)$.

Solution

$$
\begin{aligned}
&4(5 - 3)^2 + 24 \div (6 - 2) && \\
&= 4(2)^2 + 24 \div 4 && \text{Brackets} \\
&= 4(4) + 24 \div 4 && \text{Exponents/powers} \\
&= 16 + 24 \div 4 && \text{Multiplication} \\
&= 16 + 6 && \text{Division} \\
&= 22 && \text{Addition}
\end{aligned}
$$

exam focus

Remember to confirm your answer with a calculator.

Fractions

A fraction is written as two whole numbers, one over the other, separated by a bar.

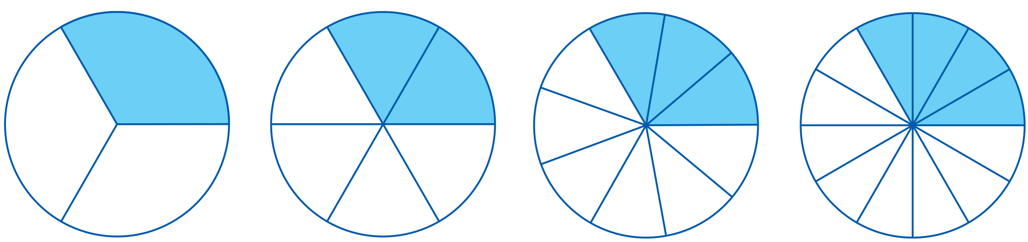
key point

$$\text{Fraction} = \frac{\text{Numerator}}{\text{Denominator}}$$

Equivalent fractions are fractions that are equal. For example:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$$

This can be shown on a diagram where the same proportion is shaded in each circle.



$\frac{1}{3}$ = $\frac{2}{6}$ $\frac{3}{9}$ = $\frac{4}{12}$

key point

The order of operations, BEMDAS, for fractions is the same as for integers.

Example

Bren is trying to subtract $\frac{1}{5}$ from $\frac{7}{8}$.

His attempt is shown here: $\frac{7}{8} - \frac{1}{5} = \frac{6}{3} = 2$

(i) Explain what Bren has done wrong.

(ii) Write out the correct solution.

Solution

(i) It seems that Bren has subtracted top from top and bottom from bottom.

key point

For subtraction or addition of fractions, we must find a common denominator.

(ii) $\frac{7}{8} - \frac{1}{5} \Rightarrow$ common denominator $= 8 \times 5 = 40$

$$\text{Then } \frac{7}{8} - \frac{1}{5} = \frac{(5)(7) - (8)(1)}{40} = \frac{35 - 8}{40} = \frac{27}{40}$$

Example

Write the following as one fraction: $\frac{10}{3}\left(\frac{3}{5} - \frac{1}{2}\right)^2 - \frac{5}{6}$

Solution

$$\frac{10}{3}\left(\frac{3}{5} - \frac{1}{2}\right)^2 - \frac{5}{6} \qquad \text{Brackets}$$

$$= \frac{10}{3}\left(\frac{(3)(2) - (1)(5)}{10}\right)^2 - \frac{5}{6}$$

$$= \frac{10}{3}\left(\frac{6 - 5}{10}\right)^2 - \frac{5}{6}$$

$$= \frac{10}{3}\left(\frac{1}{10}\right)^2 - \frac{5}{6} \qquad \text{Exponent (power/index)}$$

$$= \frac{10}{3}\left(\frac{1}{100}\right) - \frac{5}{6} \qquad \text{Multiply}$$

$$= \frac{10}{300} - \frac{5}{6}$$

$$= \frac{1}{30} - \frac{5}{6} \qquad \text{Subtract}$$

$$= \frac{(1)(1) - (5)(5)}{30}$$

$$= \frac{1 - 25}{30}$$

$$= \frac{-24}{30} \qquad \text{Simplify}$$

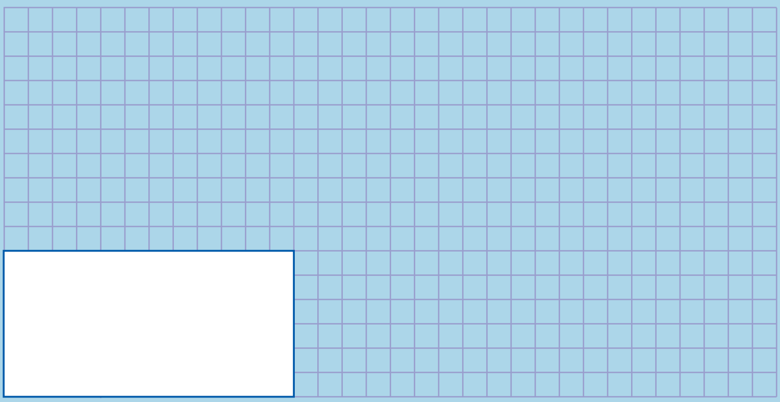
$$= -\frac{4}{5}$$

key point

Finally, reduce the fraction to its lowest terms.

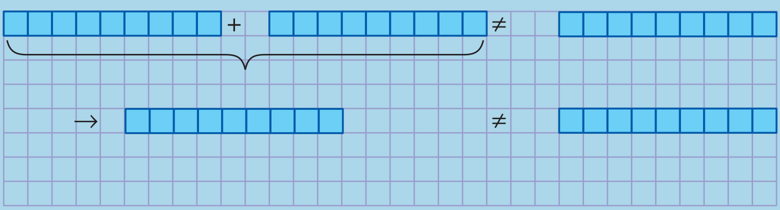
exam Q

(i) The diagram below shows three-fifths of a rectangle.
Complete the rectangle on the grid.



(ii) By shading appropriate sections of the strips below, show that

$$\frac{1}{3} + \frac{2}{6} \neq \frac{3}{9}$$



Solution

(i) By counting the rows (6) and the columns (12), the area of the given rectangle equals $6 \times 12 = 72$ square units.

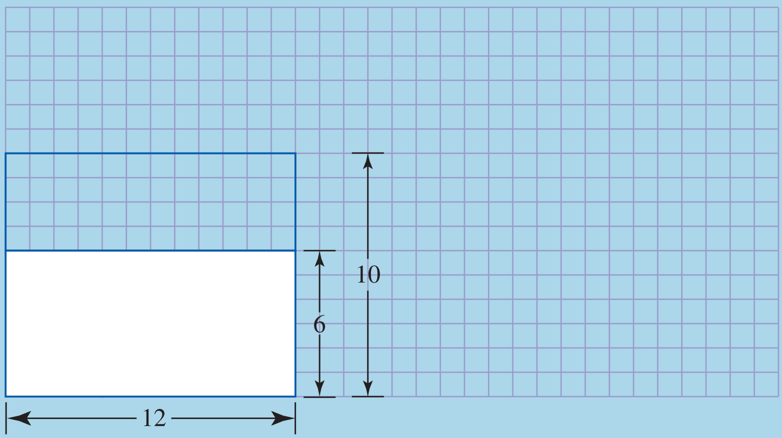
This tells us that $\frac{3}{5}$ of the rectangle $= 72$ square units.

$\Rightarrow \frac{1}{5}$ of the rectangle $= \frac{72}{3} = 24$ square units.

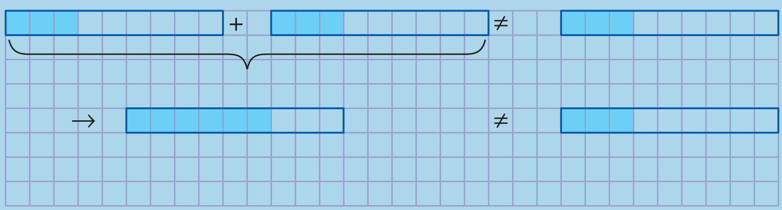
We conclude the full rectangle $= 24 \times 5 = 120$ square units.

exam focus

Many candidates simply counted 6 units (boxes) in height and since $\frac{6}{10} = \frac{3}{5}$ they wrote down the height of the full rectangle as 10 units (boxes) (see diagram).



(ii) $\frac{1}{3} + \frac{2}{6} \neq \frac{3}{9}$



key point

$\frac{2}{6} = \frac{1}{3}$